Magic Rectangles

1	36	37	72	73	108	109	144	145	180	190	207	226	243	262	279	298	315	334	351
360	325	324	289	288	253	252	217	216	181	171	154	135	118	99	82	63	46	27	10
2	35	38	71	74	107	110	143	146	179	191	206	227	242	263	278	299	314	335	350
359	326	323	290	287	254	251	218	215	182	170	155	134	119	98	83	62	47	26	11
3	34	39	70	75	106	111	142	147	178	192	205	228	241	264	277	300	313	336	349
358	327	322	291	286	255	250	219	214	183	169	156	133	120	97	84	61	48	25	12
4	33	40	69	76	105	112	141	148	177	193	204	229	240	265	276	301	312	337	348
357	328	321	292	285	256	249	220	213	184	168	157	132	121	96	85	60	49	24	13
5	32	41	68	77	104	113	140	149	176	194	203	230	239	266	275	302	311	338	347
356	329	320	293	284	257	248	221	212	185	167	158	131	122	95	86	59	50	23	14
6	31	42	67	78	103	114	139	150	175	195	202	231	238	267	274	303	310	339	346
355	330	319	294	283	258	247	222	211	186	166	159	130	123	94	87	58	51	22	15
7	30	43	66	79	102	115	138	151	174	196	201	232	237	268	273	304	309	340	345
354	331	318	295	282	259	246	223	210	187	165	160	129	124	93	88	57	52	21	16
8	29	44	65	80	101	116	137	152	173	197	200	233	236	269	272	305	308	341	344
353	332	317	296	281	260	245	224	209	188	164	161	128	125	92	89	56	53	20	17
9	28	45	64	81	100	117	136	153	172	198	199	234	235	270	271	306	307	342	343
352	333	316	297	280	261	244	225	208	189	163	162	127	126	91	90	55	54	19	18

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Contents

1 Magic Rectangles

A magic rectangle is an arrangement of the numbers from 1 to mn in a rectangle with m rows and n columns, such that the sum Z of all numbers in the rows is equal to the sum S of the numbers in the columns. Since

$$\frac{mn \cdot (mn+1)}{2}$$

holds for the sum of the first mn natural numbers, the following two formulas apply for the row and column sums.

$$Z = n \cdot \frac{mn+1}{2} \qquad S = m \cdot \frac{mn+1}{2}$$

If the product mn is even, it follows that $mn+1$ is odd. This means that the number of rows and columns must both be even so that dividing by 2 does not result in a decimal number.

However, if the product mn is odd, both m and n are odd. In this case, $mn+1$ is even, and the two expressions result in an integer when divided. It follows that in a magic rectangle, the number of rows and the number of columns are either both even or both odd. Any other configuration is impossible.

Mathematically, this fact can be formulated as follows:

A magic rectangle $R(m, n)$ exists if and only if $m, n > 1$, $mn > 4$ and $m \equiv n$ (modulo 2) applies.

Two examples of magic rectangles $R_1(3, 5)$ and $R_2(4, 8)$ are shown in figure 1.1.

5	7	4	10	14	40
13	15	8	3	1	40
6	2	12	11	9	40
24	24	24	24	24	

a) $R_1(3,5)$

16	17	12	21	8	25	4	29	132
18	15	22	11	26	7	30	3	132
19	14	23	10	27	6	31	2	132
13	20	9	24	5	28	1	32	132
66	66	66	66	66	66	66	66	

b) $R_2(4,8)$

Figure 1.1: Magic rectangles of sizes 3 x 5 and 4 x 8

Since the authors of the original articles discussed here often use matrices, they follow the usual mathematical notation and define the upper left corner (1, 1) as origin. Therefore, this book follows this notation and adjusts the coordinate system for the rectangles accordingly. The upper left corner has the coordinates (1, 1), the columns j are denoted from left to right with $1, 2, 3, \ldots$ and also the rows i from top to bottom.

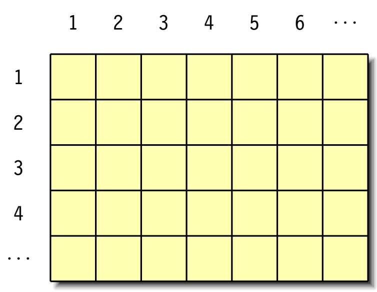


Figure 1.2: Coordinate system

Rows and columns of a magic rectangles can be permuted independently without losing the magic property. For example, in figure 1.3, the two magic rectangles $R'_1(3, 5)$ and $R'_2(4, 8)$ are shown, which are the result of permutations of the two rectangles from figure 1.1.

	1	4	5	3	2
2	13	3	1	8	15
3	6	11	9	12	2
1	5	10	14	4	7

a) $R'_1(3, 5)$

	8	7	2	1	5	6	3	4
4	32	1	20	13	5	28	9	24
1	29	4	17	16	8	25	12	21
3	2	31	14	19	27	6	23	10
2	3	30	15	18	26	7	22	11

b) $R'_2(4, 8)$

Figure 1.3: Magic rectangles with permuted rows and columns

2 Rectangles of size 2p x 2q

2.1 Trenkler - Rectangles of size 2 x 2k

For at least 1000 years, it is known how to create magic rectangles with two rows. Various Arab authors from that time used this technique to construct magic squares.

Marian Trenkler proves in an article that magic rectangles $R(2, 2k)$ exist for all $k > 1$.[1] The numbers $1, 2, \ldots, 2k$ are placed from left to right in the upper row of such a rectangle and the numbers $4k, 4k - 1, \ldots, 2k + 1$ in the row below.

1	2	3	$\cdots$	$2k-1$	$2k$
$4k$	$4k-1$	$4k-2$	$\cdots$	$2k+2$	$2k+1$

Figure 2.1: Arrangement of numbers

Thus, the column sums match, but not yet the row sums. If k is even, the numbers in columns j are swapped exactly when $j \equiv 2$ or $j \equiv 3$ (modulo 4).

1	2	3	4
1	7	6	4
8	2	3	5

a) 2 x 4 - rectangle

1	2	3	4	5	6	7	8
1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

b) 2 x 8 - rectangle

Figure 2.2: Magic rectangles 2 x 4k

There is also another arrangement possible that leads to a magic rectangle with this size. For this arrangement, simply swap the numbers in the center $2k$ columns.

1	2	14	13	12	11	7	8
16	15	3	4	5	6	10	9

Figure 2.3: Alternative arrangement

If, on the other hand, k is odd, the same procedure is followed in principle, except that in the group of the last six columns the numbers in the first and third columns of this group are swapped.

1	2	3	4	5	6
12	2	10	4	5	6
1	11	3	9	8	7

a) 2 x 6 - rectangle

1	2	3	4	5	6	7	8	9	10
1	19	18	4	16	6	14	8	9	10
20	2	3	17	5	15	7	13	12	11

group 1 (columns 1–4) group 2 (columns 5–10)

b) 2 x 10 - rectangle

Figure 2.4: Magic rectangles 2 x 4k+2

[1] Trenkler [23]

If the rectangle has even more columns, you can mix the presented variants, as can be seen in the example of a rectangle with 16 columns.

1	31	30	4	5	27	26	8	9	10	22	21	20	19	15	16
32	2	3	29	28	6	7	25	24	23	11	12	13	14	18	17

group 1 group 2 group 3

Figure 2.5: Variations in arrangement

Of course, all rectangles remain magic even if you permute the rows and columns of each subgroup independently.

2.2 Harmuth

T. Harmuth has described four systematic methods to generate magic rectangles $R(m, n)$ if the number of rows and columns is even and at least one of the two numbers is a multiple of 4.[2] However, Harmuth only gives examples for the size 8 x 8 and also speaks of magic squares, ignoring the diagonals.

Therefore, an article by Phillips should also be mentioned that gives an overview of which sizes of rectangles each method can be applied to.[3]

- method 1 and 2: the number of columns is a multiple of 4
- method 3 and 4: the number of rows is a multiple of 4
- if both, the number of rows and columns is a multiple of 4, all four methods can be used

Method 1

This method works whenever the number of columns with $n = 4k$ is a multiple of 4. For a rectangle of size 6 x 8, $\frac{m}{2} = 3$ odd numbers starting from 1 are entered from top to bottom in the first column, leaving the following row empty each time. The even numbers, on the other hand, are entered in the horizontally symmetrical column at the right margin, starting here in the second row from the top.

Exactly the opposite is done for the still empty cells in these columns. In the left column, the largest numbers from $mn = 48$ downwards and from top to bottom are entered into the empty cells. In the right column, on the other hand, the odd numbers from $mn - 1 = 47$ downwards.

1							
							2
3							
							4
5							
							6

1							47
48							2
3							45
46							4
5							43
44							6

Figure 2.6: Filling the two outer columns

2 Harmuth [11]
3 Phillips [16]

The procedure for the next two columns is different. You start in the bottom row of the penultimate column and enter the next odd numbers from 7 upwards from the bottom to the top, skipping one row as usual. Then the even numbers follow in the second column, where you start in the second row from the bottom.

The next large even numbers not yet used are entered from 42 downwards in the penultimate column from bottom to top in the cells that are still empty. Likewise, the corresponding odd numbers are entered from 41 downwards and from bottom to top in the second column.

1	12						47
48						11	2
3	10						45
46						9	4
5	8						43
44						7	6

1	12					38	47
48	37					11	2
3	10					40	45
46	39					9	4
5	8					42	43
44	41					7	6

Figure 2.7: Filling the two adjacent columns

Continue in this way, always alternating, until the magic rectangle is created, shown in figure 2.8.

1	12	13	24	26	35	38	47
48	37	36	25	23	14	11	2
3	10	15	22	28	33	40	45
46	39	34	27	21	16	9	4
5	8	17	20	30	31	42	43
44	41	32	29	19	18	7	6

Figure 2.8: Method 1 - magic rectangle of size 6×8 (Harmuth)

In another example, figure 2.9 shows the process to create a magic rectangle of size 10×12.

1	20	21	40	41	60	62	79	82	99	102	119
120	101	100	81	80	61	59	42	39	22	19	2
3	18	23	38	43	58	64	77	84	97	104	117
118	103	98	83	78	63	57	44	37	24	17	4
5	16	25	36	45	56	66	75	86	95	106	115
116	105	96	85	76	65	55	46	35	26	15	6
7	14	27	34	47	54	68	73	88	93	108	113
114	107	94	87	74	67	53	48	33	28	13	8
9	12	29	32	49	52	70	71	90	91	110	111
112	109	92	89	72	69	51	50	31	30	11	10

Figure 2.9: Method 1 - magic rectangle of size 10×12 (Harmuth)

Method 2

Also in this method, the number of columns with $n = 4k$ must be a multiple of 4. This method is similar to the first method. Here, however, a distinction is no longer made between even and odd numbers when entering the numbers. Instead, consecutive numbers are always selected.

For a rectangle of size 6×8, the first $\frac{m}{2} = 3$ numbers are again entered from top to bottom in the left column, skipping one row as usual. Then the next numbers follow in the right column, again starting in the second row. In the next step, the large numbers from $mn = 48$ downwards and from top to bottom follow in the left column, with the numbers decreasing with each step. Likewise, the still empty cells in the right column are filled up with the next numbers starting from 45.

1							
							4
2							
							5
3							
							6

1							45
48							4
2							44
47							5
3							43
46							6

Figure 2.10: Filling the two outer columns

For the next two columns, the procedure is different again, as in method 1. You start in the lower row of the penultimate column and enter the next not yet used numbers from 7 upwards from the bottom upwards, skipping one row as usual. Then the next numbers from 10 follow in the second column, where you start in the second row from the bottom.

Then, the next numbers not yet used are entered, starting with the largest of these numbers with 42 and always decreasing the following numbers. Start in the penultimate column and enter the numbers from bottom to top in the still empty cells. Proceed in the same way with the numbers from 39 in the second column.

1	12						45
48						9	4
2	11						44
47						8	5
3	10						43
46						7	6

1	12					40	45
48	37					9	4
2	11					41	44
47	38					8	5
3	10					42	43
46	39					7	6

Figure 2.11: Filling the two adjacent columns

This scheme is continued until the magic rectangle of figure 2.12 is created.

1	12	13	24	28	33	40	45
48	37	36	25	21	16	9	4
2	11	14	23	29	32	41	44
47	38	35	26	20	17	8	5
3	10	15	22	30	31	42	43
46	39	34	27	19	18	7	6

Figure 2.12: Method 2 - magic rectangle of size 6×8 (Harmuth)

As a second example, a magic rectangle of size 10 x 12 is given, which was also constructed using this method and is shown in figure 2.13.

1	20	21	40	41	60	66	75	86	95	106	115
120	101	100	81	80	61	55	46	35	26	15	6
2	19	22	39	42	59	67	74	87	94	107	114
119	102	99	82	79	62	54	47	34	27	14	7
3	18	23	38	43	58	68	73	88	93	108	113
118	103	98	83	78	63	53	48	33	28	13	8
4	17	24	37	44	57	69	72	89	92	109	112
117	104	97	84	77	64	52	49	32	29	12	9
5	16	25	36	45	56	70	71	90	91	110	111
116	105	96	85	76	65	51	50	31	30	11	10

Figure 2.13: Method 2 - magic rectangle of size 10 x 12 (Harmuth)

The magic rectangles created with this method are always concentric, although this property here refers to a double frame. Three other magic rectangles of sizes 10 x 12, 6 x 8 and 2 x 4 are embedded in the magic rectangle from figure 2.14. The row sums of the rectangles are 1800, 1350, 900 and 450 and the column sums are 1575, 1125, 675 and 225.

1	28	29	56	57	84	85	112	120	133	148	161	176	189	204	217
224	197	196	169	168	141	140	113	105	92	77	64	49	36	21	8
2	27	30	55	58	83	86	111	121	132	149	160	177	188	205	216
223	198	195	170	167	142	139	114	104	93	76	65	48	37	20	9
3	26	31	54	59	82	87	110	122	131	150	159	178	187	206	215
222	199	194	171	166	143	138	115	103	94	75	66	47	38	19	10
4	25	32	53	60	81	88	109	123	130	151	158	179	186	207	214
221	200	193	172	165	144	137	116	102	95	74	67	46	39	18	11
5	24	33	52	61	80	89	108	124	129	152	157	180	185	208	213
220	201	192	173	164	145	136	117	101	96	73	68	45	40	17	12
6	23	34	51	62	79	90	107	125	128	153	156	181	184	209	212
219	202	191	174	163	146	135	118	100	97	72	69	44	41	16	13
7	22	35	50	63	78	91	106	126	127	154	155	182	183	210	211
218	203	190	175	162	147	134	119	99	98	71	70	43	42	15	14

Figure 2.14: Method 2 - concentric magic rectangle of size 14 x 16 (Harmuth)

Method 3

This method can be applied to a rectangle $R(m, n)$ if the number of rows with $m = 4k$ is a multiple of

4. Start with a rectangle of size 8 × 10 by entering the m odd numbers starting from 1 from top to bottom in the two outer columns. The first $\frac{m}{4} = 2$ numbers in the left column, followed $\frac{m}{2} = 4$ numbers in the right column. The end is formed by further $\frac{m}{4}$ numbers, again in the left column. This is followed by the even numbers in the two neighboring columns, as shown in figure 2.15a.

Now, as can be seen in figure 2.15b, the even numbers follow from $mn = 80$ downwards. First again $\frac{m}{4}$ numbers from top to bottom in the right column, followed by $\frac{m}{2}$ numbers in the left column and finally again $\frac{m}{4}$ numbers in the right column. The odd numbers from $mn - 1 = 79$ on are entered accordingly, only starting at the top of the penultimate column.

1	2								
3	4								
								6	5
								8	7
								10	9
								12	11
13	14								
15	16								

a) small numbers

1	2							79	80
3	4							77	78
76	75							6	5
74	73							8	7
72	71							10	9
70	69							12	11
13	14							67	68
15	16							65	66

b) greater numbers

Figure 2.15: Entering the numbers in the two outer columns

Continue in this way with the following columns, always starting with the smallest or largest numbers that have not yet been entered. Since 16 numbers have already been entered so far, the next columns are started with 17 for the odd numbers and with 18 for the even numbers. Accordingly, the two large numbers are 48 and 47 respectively.

1	2	17	18					79	80
3	4	19	20					77	78
76	75					22	21	6	5
74	73					24	23	8	7
72	71					26	25	10	9
70	69					28	27	12	11
13	14	29	30					67	68
15	16	31	32					65	66

1	2	17	18			63	64	79	80
3	4	19	20			61	62	77	78
76	75	60	59			22	21	6	5
74	73	58	57			24	23	8	7
72	71	56	55			26	25	10	9
70	69	54	53			28	27	12	11
13	14	29	30			51	52	67	68
15	16	31	32			49	50	65	66

Figure 2.16: Filling the next four columns

Now the rectangle is either already filled, or two empty columns remain, as in this example. Since only the middle $2m = 16$ numbers remain, these columns can also be filled according to the previous scheme, but here without the neighboring columns. So, the odd numbers from 33 upwards and from top to bottom and the even numbers from 48 downwards into the empty cells. The result is the magic rectangle from figure 2.17.

1	2	17	18	33	48	63	64	79	80
3	4	19	20	35	46	61	62	77	78
76	75	60	59	44	37	22	21	6	5
74	73	58	57	42	39	24	23	8	7
72	71	56	55	40	41	26	25	10	9
70	69	54	53	38	43	28	27	12	11
13	14	29	30	45	36	51	52	67	68
15	16	31	32	47	34	49	50	65	66

Figure 2.17: Method 3 - magic rectangle of size 8 x 10 (Harmuth)

Method 4

The number of rows must be a multiple of 4 again. The rectangle is filled similarly to method 3, but the distinction between the odd and even numbers is omitted. Instead, the consecutive numbers are simply always entered in the corresponding columns. Thus, the numbers 1 to 8 are first entered in the two outer columns starting in the upper left corner and, starting from the upper right corner, the numbers from $mn = 80$ downwards.

1									
2									
									3
									4
									5
									6
7									
8									

1									80
2									79
78									3
77									4
76									5
75									6
7									74
8									73

Figure 2.18: Filling the two outer columns

The next pass then starts with the number 9 in the top row of the second column and the number 72 from the upper right corner down.

1	9								80
2	10								79
78								11	3
77								12	4
76								13	5
75								14	6
7	15								74
8	16								73

1	9							72	80
2	10							71	79
78	70							11	3
77	69							12	4
76	68							13	5
75	67							14	6
7	15							66	74
8	16							65	73

Figure 2.19: Filling the two adjacent columns

Finally, two columns remain empty, which are filled accordingly. This creates the magic rectangle shown in figure 2.20.

1	9	17	25	33	48	56	64	72	80
2	10	18	26	34	47	55	63	71	79
78	70	62	54	46	35	27	19	11	3
77	69	61	53	45	36	28	20	12	4
76	68	60	52	44	37	29	21	13	5
75	67	59	51	43	38	30	22	14	6
7	15	23	31	39	42	50	58	66	74
8	16	24	32	40	41	49	57	65	73

Figure 2.20: Method 4 - magic rectangle of size 8 x 10 (Harmuth)

Example 8x12

If the number of rows and columns are both multiples of 4, all four methods can be used. In the first example, a magic rectangle of size 8 x 12 is created using method 1. The result is shown in figure 2.21.

1	16	17	32	33	48	50	63	66	79	82	95
96	81	80	65	64	49	47	34	31	18	15	2
3	14	19	30	35	46	52	61	68	77	84	93
94	83	78	67	62	51	45	36	29	20	13	4
5	12	21	28	37	44	54	59	70	75	86	91
92	85	76	69	60	53	43	38	27	22	11	6
7	10	23	26	39	42	56	57	72	73	88	89
90	87	74	71	58	55	41	40	25	24	9	8

Figure 2.21: Method 1 - magic rectangle of size 8 x 12 (Harmuth)

A second magic rectangle of this size was created using method 3. This rectangle can be seen in figure 2.22 .

1	2	17	18	33	34	63	64	79	80	95	96
3	4	19	20	35	36	61	62	77	78	93	94
92	91	76	75	60	59	38	37	22	21	6	5
90	89	74	73	58	57	40	39	24	23	8	7
88	87	72	71	56	55	42	41	26	25	10	9
86	85	70	69	54	53	44	43	28	27	12	11
13	14	29	30	45	46	51	52	67	68	83	84
15	16	31	32	47	48	49	50	65	66	81	82

Figure 2.22: Method 3 - magic rectangle of size 8 x 12 (Harmuth)

2.3 Phillips

J. P. N. Phillips has developed a method to generate magic rectangles $R(m, n)$ if the number of rows and columns is even. However, at least one of the two numbers must be divisible by four.[4]

His method shall first be illustrated for a rectangle of size 4×10, since the number of rows chosen here is important for later extensions to larger rectangles. Phillips starts in the upper left corner and enters the consecutive numbers from 1 diagonally to the lower right. Once you have reached the bottom, you continue diagonally upwards to the right from the next column. This movement alternates until you have reached the right edge after $n = 10$ numbers.

Then you change the diagonal movements and continue with the next n numbers in the vertically symmetrically lying cells. After one pass, $2n$ numbers are already entered, with the last number placed in the lower left corner.

1							8	9	
	2					7			10
		3			6				
			4	5					

1			17	16			8	9	
	2	18			15	7			10
	19	3			6	14			11
20			4	5			13	12	

Figure 2.23: Entering the first $2n = 20$ numbers

Further $2n$ numbers are entered similarly. The initial number is $mn + 1 - 2n = 41 - 20 = 21$ and matches in this special case with the successor of the last entered number. However, if the rectangle has more than four rows, these numbers do not match anymore, as can be seen in two following examples with $m > 4$.

But this this time, you start in the lower right corner and first move diagonally to the upper left. Thus, after these two passes, the magic rectangle of size 4×10 from figure 2.24 has been created.

1			17	16	25	24	8	9	
	2	18		26	15	7	23		10
30	19	3	27		6	14		22	11
20	29	28	4	5			13	12	21

1	32	33	17	16	25	24	8	9	40
31	2	18	34	26	15	7	23	39	10
30	19	3	27	35	6	14	38	22	11
20	29	28	4	5	36	37	13	12	21

Figure 2.24: Magic rectangle of size 4×10 (Phillips)

Example 4x8

For a 4×8 rectangle, you start accordingly. However, after the first $n = 8$ numbers you have reached the upper right corner, so that you must continue for the vertically symmetrical cells in the lower right corner. Thus, the rectangle R_1 is filled as in figure 2.25.

[4] Phillips [17]

1							8
	2					7	
		3			6		
			4	5			

1			13	12			8
	2	14			11	7	
	15	3			6	10	
16			4	5			9

Figure 2.25: Entering the first $2n = 16$ numbers

Since the lower right corner is now already occupied, you must proceed differently for the next $2n = 16$ numbers. So, you have to move the rows of the rectangle cyclically down by two rows.

↓

	15	3			6	10	
16			4	5			9
1			13	12			8
	2	14			11	7	

Figure 2.26: Shift all rows cyclically

Now the upper left corner is empty again, and you can enter the next numbers as it was done in the first pass for the rectangle R_1. Thus, a magic rectangle has been created for this size as well, which is shown in figure 2.27.

17	15	3			6	10	24
16	18		4	5		23	9
1		19	13	12	22		8
	2	14	20	21	11	7	

17	15	3	29	28	6	10	24
16	18	30	4	5	27	23	9
1	31	19	13	12	22	26	8
32	2	14	20	21	11	7	25

Figure 2.27: Magic rectangle of size 4 x 8 (Phillips)

Example 4x12

Another special case occurs when the number of columns is an odd multiple of 4, i.e. 12, 20, 28, ... Here the row sums cannot be balanced appropriately, so the balance must be ensured differently.

First, you fill the rectangle with two passes as usual. The numbers in the four columns at the right edge are actually unimportant. They are only used so that $2n$ numbers are entered in each pass, which are important for the arrangement of the numbers in the remaining columns. These numbers at the right margin will be arranged differently later.

1			21	20			8	9			13
	2	22			19	7			10	14	
	23	3			6	18			15	11	
24			4	5			17	16			12

a) entering the first 24 numbers

	23	3			6	18			15	11	
24			4	5			17	16			12
1			21	20			8	9			13
	2	22			19	7			10	14	

b) shift all rows

25	23	3	45	44	6	18	32	33	15	11	37
24	26	46	4	5	43	31	17	16	34	38	12
1	47	27	21	20	30	42	8	9	39	35	13
48	2	22	28	29	19	7	41	40	10	14	36

c) entering the remaining 24 numbers

25	23	3	45	44	6	18	32	33	15	11	37
24	26	46	4	5	43	31	17	16	34	38	12
1	47	27	21	20	30	42	8	9	39	35	13
48	2	22	28	29	19	7	41	40	10	14	36

R_8 R_r

d) unsuitable subsquare at the right edge

Figure 2.28: Preliminary entering of numbers into a 4 x 12 - rectangle

While the left eight columns in figure 2.28d result in a rectangle R_8 with row sums 196 and column sums 98, the 4 x 4 rectangle R_r on the right edge has different row sums 100 and 96. This is because the existing numbers $9, 10, \ldots, 16$ and $33, 34, \ldots, 40$ were not arranged appropriately with this number of columns.

If, on the other hand, you take the magic rectangle $R_4(4, 4)$ and increase the numbers that are none or equal to 8 by 8 and the other numbers by 24, you obtain the rectangle R'_4.

1	11	6	16
12	2	15	5
8	14	3	9
13	7	10	4

9	35	14	40
36	10	39	13
16	38	11	33
37	15	34	12

Figure 2.29: Rectangle R_4 and suitable rectangle R'_4

With this construction, the rectangle R'_4 contains the same numbers as the original rectangle R_r at the right edge and with 98 additionally also the same row and column sums. If you now replace the four columns at the right edge by the columns of R'_4, you get the magic rectangle from figure 2.30.

25	23	3	45	44	6	18	32	9	35	14	40
24	26	46	4	5	43	31	17	36	10	39	13
1	47	27	21	20	30	42	8	16	38	11	33
48	2	22	28	29	19	7	41	37	15	34	12

Figure 2.30: Magic rectangle of size 4 x 12 (Phillips)

Example 8x10

If the number of rows is a multiple of 4, several 4 x n rectangles are inserted one above the other into the target rectangle. Thus, if a 8 x 10 rectangle is to be created, a 4 x 10 rectangle is constructed first. The first $2n = 20$ numbers begin with initial number 1 and are entered as usual. The initial number of the second sequence of 20 numbers, which begins in the lower right corner, is now $mn + 1 - 2n$ „ i.e. 61. This is the only way to ensure that the column sums are all the same later in the target rectangle.

1	72	73	17	16	65	64	8	9	80
71	2	18	74	66	15	7	63	79	10
70	19	3	67	75	6	14	78	62	11
20	69	68	4	5	76	77	13	12	61

Figure 2.31: First rectangle of size 4 x 10

Now a second rectangle of size 4 x 10 is constructed. The initial number from the first rectangle increases by $2n = 20$ to 21, while the second initial number decreases by $2n$ from 61 to 41.

21	52	53	37	36	45	44	28	29	60
51	22	38	54	46	35	27	43	59	30
50	39	23	47	55	26	34	58	42	31
40	49	48	24	25	56	57	33	32	41

Figure 2.32: Second rectangle of size 4 x 10

Both rectangles are now entered one above the other in the target rectangle. The resulting magic rectangle of size 8 x 10 is shown in figure 2.33.

21	52	53	37	36	45	44	28	29	60
51	22	38	54	46	35	27	43	59	30
50	39	23	47	55	26	34	58	42	31
40	49	48	24	25	56	57	33	32	41
1	72	73	17	16	65	64	8	9	80
71	2	18	74	66	15	7	63	79	10
70	19	3	67	75	6	14	78	62	11
20	69	68	4	5	76	77	13	12	61

Figure 2.33: Magic rectangle of size 8 x 10 (Phillips)

Since all 4 x 10 rectangles used have the same row and column sums, you can permute the rows and columns of these rectangles independently to create other magic rectangles.

Example 6x8

Proceed similarly if the number of rows is not a multiple of 4. First, rectangles of size 4 x n are formed, which are inserted one above the other into the target rectangle, as in the example of size 8 x 10. For a 6 x 8 rectangle, only one auxiliary rectangle of size 4 x 8 is required, which is shown in figure 2.34. As initial numbers 1 and $mn + 1 - 2n = 49 - 16 = 33$ were used here.

33	15	3	45	44	6	10	40
16	34	46	4	5	43	39	9
1	47	35	13	12	38	42	8
48	2	14	36	37	11	7	41

33	15	3	45	44	6	10	40
16	34	46	4	5	43	39	9
1	47	35	13	12	38	42	8
48	2	14	36	37	11	7	41

Figure 2.34: Rectangle of size 4 x 8

Now only the upper two rows of the target rectangle are empty and the middle $2n$ numbers $17, 18, \ldots, 32$ are missing. Therefore, a magic 2 x n rectangle is created and the numbers of this rectangle are increased by the corresponding value for the needed numbers. In this case, all numbers were increased by 16 because the numbers $17, 18, \ldots, 32$ are still missing in the target rectangle which was partially filled so far.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

a) rectangle of size 2 x 10

17	31	30	20	21	27	26	24
32	18	19	29	28	22	23	25

b) rectangle with increased numbers

If this rectangle is inserted into the empty area of the target rectangle, the magic rectangle from figure 2.35 is created.

17	31	30	20	21	27	26	24
32	18	19	29	28	22	23	25
33	15	3	45	44	6	10	40
16	34	46	4	5	43	39	9
1	47	35	13	12	38	42	8
48	2	14	36	37	11	7	41

Figure 2.35: Magic rectangle of size 6 x 8 (Phillips)

2.4 De Los Reyes - Das - Midha - Vellaisamy

In the method of De Los Reyes, Das, Midha and Vellaisamy for the construction of magic rectangles with an even number of rows, two cases are distinguished, where always $m = 2p$ and $n = 2q$ applies.[5] In the first case, at least one of p and q is even. Here it is assumed that p is even. Otherwise, you can simply work with the transposed matrix and undo this transformation with the result matrix.

First, the numbers $1, 2, \ldots, m$ are entered from top to bottom in the first column of a matrix. Then the numbers $m+1, m+2, \ldots, 2m$ follow from bottom to top in the second column, followed by $2m+1, 2m+2, \ldots, 3m$ in the third column and so on. Thus, the matrix is filled alternately from top to bottom and from bottom to top with consecutive numbers $1, 2, \ldots, mn$.

[5] Midha - Das - Midha - Vellaisamy [15]

$$\begin{pmatrix} 1 & 2m & 2m+1 & 4m & \dots \\ 2 & 2m-1 & 2m+2 & 4m-1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ m-1 & m+2 & 3m-1 & 3m+2 & \dots \\ m & m+1 & 3m & 3m+1 & \dots \end{pmatrix}$$

Such an arrangement of numbers is also called an *Serpentine-Matrix* and has been transferred to a rectangle in figure 2.36.

↓		↓		↓		↓		↓	
1	16	17	32	33	48	49	64	65	80
2	15	18	31	34	47	50	63	66	79
3	14	19	30	35	46	51	62	67	78
4	13	20	29	36	45	52	61	68	77
5	12	21	28	37	44	53	60	69	76
6	11	22	27	38	43	54	59	70	75
7	10	23	26	39	42	55	58	71	74
8	9	24	25	40	41	56	57	72	73
	↑		↑		↑		↑		↑

Figure 2.36: Rectangle from a serpentine matrix

If you now reverse the center p rows, this rectangle becomes the magic 8×10 rectangle from figure 2.37.

1	16	17	32	33	48	49	64	65	80
2	15	18	31	34	47	50	63	66	79
78	67	62	51	46	35	30	19	14	3
77	68	61	52	45	36	29	20	13	4
76	69	60	53	44	37	28	21	12	5
75	70	59	54	43	38	27	22	11	6
7	10	23	26	39	42	55	58	71	74
8	9	24	25	40	41	56	57	72	73

Figure 2.37: Magic rectangle of size 8×10 (De Los Reyes et al.)

Example 6x10

In the second example, a magic rectangle of size 6×10 shall be created. In this case, both $p = 3$ and $q = 5$ are odd, and the construction becomes a bit more extensive. You start again with the serpentine matrix.

↓		↓		↓		↓		↓	
1	12	13	24	25	36	37	48	49	60
2	11	14	23	26	35	38	47	50	59
3	10	15	22	27	34	39	46	51	58
4	9	16	21	28	33	40	45	52	57
5	8	17	20	29	32	41	44	53	56
6	7	18	19	30	31	42	43	54	55
	↑		↑		↑		↑		↑

Figure 2.38: Rectangle from a serpentine matrix

Then the left $2 \cdot \left\lfloor \frac{q}{2} \right\rfloor = 4$ columns are considered, which will be reversed.

6	7	18	19	25	36	37	48	49	60
5	8	17	20	26	35	38	47	50	59
4	9	16	21	27	34	39	46	51	58
3	10	15	22	28	33	40	45	52	57
2	11	14	23	29	32	41	44	53	56
1	12	13	24	30	31	42	43	54	55

Figure 2.39: Step 2 - reverse columns

In the third step, for the columns $1 \leq j \leq 2 \cdot \left\lfloor \frac{q}{2} \right\rfloor$ any p numbers from column j are swapped with the numbers from their horizontally symmetric column. However, the two middle columns remain unchanged. For simplicity, the upper p numbers are chosen in this example.

60	49	48	37	25	36	19	18	7	6
59	50	47	38	26	35	20	17	8	5
58	51	46	39	27	34	21	16	9	4
3	10	15	22	28	33	40	45	52	57
2	11	14	23	29	32	41	44	53	56
1	12	13	24	30	31	42	43	54	55

Figure 2.40: Step 3 - Swap numbers

In step 4, the middle $p - 3$ numbers in the two center columns are swapped with each other. However, this case does not occur with this rectangle with $p = 3$. In the following example for a rectangle of the size 10 x 14, however, two numbers are swapped with each other in this step.

In addition, there is a second swap in these two columns. This affects the rows $t + \frac{p-3}{2}$ for $t = 1$ and $t = 3$. In this example, with $p = 3$, these are rows 1 and 3. With these swaps, the magic rectangle of size 6 x 10 from figure 2.41 is obtained. The row and column sums of this rectangle are 305 and 183, respectively.

60	49	48	37	36	25	19	18	7	6
59	50	47	38	26	35	20	17	8	5
58	51	46	39	34	27	21	16	9	4
3	10	15	22	28	33	40	45	52	57
2	11	14	23	29	32	41	44	53	56
1	12	13	24	30	31	42	43	54	55

Figure 2.41: Magic rectangle of size 6 x 10 (De Los Reyes et al.)

Example 10x14

This procedure shall be illustrated once again with a rectangle of the size 10 x 14 because in the first example a partial step did not have to be carried out because of the smaller size. Start again as in figure 2.42 with the rectangle from a serpentine matrix.

↓		↓		↓		↓		↓		↓		↓	
1	20	21	40	41	60	61	80	81	100	101	120	121	140
2	19	22	39	42	59	62	79	82	99	102	119	122	139
3	18	23	38	43	58	63	78	83	98	103	118	123	138
4	17	24	37	44	57	64	77	84	97	104	117	124	137
5	16	25	36	45	56	65	76	85	96	105	116	125	136
6	15	26	35	46	55	66	75	86	95	106	115	126	135
7	14	27	34	47	54	67	74	87	94	107	114	127	134
8	13	28	33	48	53	68	73	88	93	108	113	128	133
9	12	29	32	49	52	69	72	89	92	109	112	129	132
10	11	30	31	50	51	70	71	90	91	110	111	130	131
	↑		↑		↑		↑		↑		↑		↑

Figure 2.42: Rectangle from a serpentine matrix

Then, in the second step, reverse the left $2 \cdot \lfloor \frac{q}{2} \rfloor = 6$ rows. (see figure 2.43)

10	11	30	31	50	51	61	80	81	100	101	120	121	140
9	12	29	32	49	52	62	79	82	99	102	119	122	139
8	13	28	33	48	53	63	78	83	98	103	118	123	138
7	14	27	34	47	54	64	77	84	97	104	117	124	137
6	15	26	35	46	55	65	76	85	96	105	116	125	136
5	16	25	36	45	56	66	75	86	95	106	115	126	135
4	17	24	37	44	57	67	74	87	94	107	114	127	134
3	18	23	38	43	58	68	73	88	93	108	113	128	133
2	19	22	39	42	59	69	72	89	92	109	112	129	132
1	20	21	40	41	60	70	71	90	91	110	111	130	131

Figure 2.43: Step 2 - reverse columns

Then follows step 3 by swapping $p = 5$ numbers in columns $1 \leq j \leq 2 \cdot \left\lfloor \frac{q}{2} \right\rfloor$. (see figure 2.44)

140	121	120	101	100	81	61	80	51	50	31	30	11	10
139	122	119	102	99	82	62	79	52	49	32	29	12	9
138	123	118	103	98	83	63	78	53	48	33	28	13	8
137	124	117	104	97	84	64	77	54	47	34	27	14	7
136	125	116	105	96	85	65	76	55	46	35	26	15	6
5	16	25	36	45	56	66	75	86	95	106	115	126	135
4	17	24	37	44	57	67	74	87	94	107	114	127	134
3	18	23	38	43	58	68	73	88	93	108	113	128	133
2	19	22	39	42	59	69	72	89	92	109	112	129	132
1	20	21	40	41	60	70	71	90	91	110	111	130	131

Figure 2.44: Step 3 - swap numbers

Now follows the fourth step, which did not have to be carried out in the previous example. Here, the middle $p - 3 = 2$ numbers in the two center columns are now swapped with each other. (see figure 2.45)

140	121	120	101	100	81	61	80	51	50	31	30	11	10
139	122	119	102	99	82	62	79	52	49	32	29	12	9
138	123	118	103	98	83	63	78	53	48	33	28	13	8
137	124	117	104	97	84	64	77	54	47	34	27	14	7
136	125	116	105	96	85	76	65	55	46	35	26	15	6
5	16	25	36	45	56	75	66	86	95	106	115	126	135
4	17	24	37	44	57	67	74	87	94	107	114	127	134
3	18	23	38	43	58	68	73	88	93	108	113	128	133
2	19	22	39	42	59	69	72	89	92	109	112	129	132
1	20	21	40	41	60	70	71	90	91	110	111	130	131

Figure 2.45: Step 4 - swap numbers in the two center columns

The final swap of two pairs of numbers creates the magic rectangle from figure 2.46. The row and column sums of this rectangle are 595 and 255, respectively.

140	121	120	101	100	81	61	80	51	50	31	30	11	10
139	122	119	102	99	82	79	62	52	49	32	29	12	9
138	123	118	103	98	83	63	78	53	48	33	28	13	8
137	124	117	104	97	84	77	64	54	47	34	27	14	7
136	125	116	105	96	85	76	65	55	46	35	26	15	6
5	16	25	36	45	56	75	66	86	95	106	115	126	135
4	17	24	37	44	57	67	74	87	94	107	114	127	134
3	18	23	38	43	58	68	73	88	93	108	113	128	133
2	19	22	39	42	59	69	72	89	92	109	112	129	132
1	20	21	40	41	60	70	71	90	91	110	111	130	131

Figure 2.46: Magic rectangle of size 10 x 14 (De Los Reyes et al.)

2.5 Bier - Kleinschmidt

m and n even and m a multiple of 4 (double-even)

The fundamental base for the methods presented in this section are centrally symmetric rectangles, whose definition was given by Bier and Kleinschmidt.[6] I reduce their very theoretical article here to the points, which are necessary for the construction of magic rectangles.

A rectangle R is called *centrally symmetric*, when its numbers can be represented as $\pm[1, 2, \ldots, x]$ and all row and column sums are 0. In their theoretical article, they prove that a centrally symmetric rectangle exists if and only if there also exists a magic rectangle of the same size.

For their proof of centrally symmetric rectangles $C(m, n)$ with an even number of rows and columns, they assume the existence of such a rectangle $C(4, n)$. They give a formula that can be used to create a rectangle with four rows and n columns. With the origin $(1, 1)$ in the upper left corner of the coordinate system, they define the numbers of the rectangle $C(4, n)$ with $n = 2s$, $1 \leq i \leq 4$, $j = 2t + r$ and $r \in \{1, 2\}$.

$$c_{ij} = \begin{cases} -4\,t - i & \text{for } |5 - 2i| = 5 - 2r \\ 4\,t + i & \text{for } |5 - 2i| = 2r - 1 \end{cases}$$

This formula yields, for example, the centrally symmetrical rectangle $C(4, 6)$. Adding 12 to all positive numbers and 13 to the negative numbers gives the magic rectangle from figure 2.47, with row sums 75 and column sums 50.

[6] Bier - Kleinschmidt [2]

-1	1	-5	5	-9	9
2	-2	6	-6	10	-10
3	-3	7	-7	11	-11
-4	4	-8	8	-12	12

a) $C(4,6)$

12	13	8	17	4	21
14	11	18	7	22	3
15	10	19	6	23	2
9	16	5	20	1	24

b) increased numbers

Figure 2.47: Magic rectangle of size 4 x 6 (Bier - Kleinschmidt)

In general, the positive numbers must be increased by $\frac{m \cdot n}{2}$ and the negative ones by $\frac{m \cdot n}{2} + 1$. Thus, for the centrally symmetric rectangle $C(4,8)$, the increments are 16 and 17. The two rectangles are shown in figure 2.48. The row sums in the magic rectangle are 132 and the column sums are 66.

-1	1	-5	5	-9	9	-13	13
2	-2	6	-6	10	-10	14	-14
3	-3	7	-7	11	-11	15	-15
-4	4	-8	8	-12	12	-16	16

a) $C(4,8)$

16	17	12	21	8	25	4	29
18	15	22	11	26	7	30	3
19	14	23	10	27	6	31	2
13	20	9	24	5	28	1	32

b) increased numbers

Figure 2.48: Magic rectangle of size 4 x 8 (Bier - Kleinschmidt)

In figure 2.49 the centrally symmetric rectangle $C(4,12)$ with 12 columns is shown.

-1	1	-5	5	-9	9	-13	13	-17	17	-21	21
2	-2	6	-6	10	-10	14	-14	18	-18	22	-22
3	-3	7	-7	11	-11	15	-15	19	-19	23	-23
-4	4	-8	8	-12	12	-16	16	-20	20	-24	24

Figure 2.49: Centrally symmetric rectangle $C(4,12)$

Converting this rectangle as described results in the magic rectangle from figure 2.50 with row sums 294 and column sums 98.

24	25	20	29	16	33	12	37	8	41	4	45
26	23	30	19	34	15	38	11	42	7	46	3
27	22	31	18	35	14	39	10	43	6	47	2
21	28	17	32	13	36	9	40	5	44	1	48

Figure 2.50: Magic rectangle of size 4 x 12 (Bier - Kleinschmidt)

This construction principle can easily be extended to rectangles where the number of rows is a multiple of 4. For these sizes, the numbers in the centrally symmetric rectangle must be expanded as specified by

Bier and Kleinschmidt with their formula for four rows. Of course, the same arrangement of signs must be considered. For eight rows, the rectangle $C(8, 12)$ is given by

-1	1	-9	9	-17	17	-25	25	-33	33	-41	41
2	-2	10	-10	18	-18	26	-26	34	-34	42	-42
3	-3	11	-11	19	-19	27	-27	35	-35	43	-43
-4	4	-12	12	-20	20	-28	28	-36	36	-44	44
-5	5	-13	13	-21	21	-29	29	-37	37	-45	45
6	-6	14	-14	22	-22	30	-30	38	-38	46	-46
7	-7	15	-15	23	-23	31	-31	39	-39	47	-47
-8	8	-16	16	-24	24	-32	32	-40	40	-48	48

Figure 2.51: Centrally symmetric rectangle $C(8, 12)$

To get the corresponding magic rectangle $R(8, 12)$ from figure 2.52, the numbers only have to be increased by $\frac{m \cdot n}{2} = 48$ or $\frac{m \cdot n}{2} + 1 = 49$, respectively. Thus, this rectangle has the row sums 582 and the column sums 388.

48	49	40	57	32	65	24	73	16	81	8	89
50	47	58	39	66	31	74	23	82	15	90	7
51	46	59	38	67	30	75	22	83	14	91	6
45	52	37	60	29	68	21	76	13	84	5	92
44	53	36	61	28	69	20	77	12	85	4	93
54	43	62	35	70	27	78	19	86	11	94	3
55	42	63	34	71	26	79	18	87	10	95	2
41	56	33	64	25	72	17	80	9	88	1	96

Figure 2.52: Magic rectangle of size 8 x 12 (Bier - Kleinschmidt)

m single-even, n>=4 even

If the number m of rows is not a multiple of 4, i.e. single-even, some additional steps are required to create a magic rectangle $R(m, n)$. First, a centrally symmetric rectangle $C(m-2, n)$ is created using the method already described and then converted into a magic rectangle $R_1(m-2, n)$. For a rectangle of size 6 x 8 with $m = 6$, the subrectangle $R_1(4, 8)$ is shown in figure 2.53.

-1	1	-5	5	-9	9	-13	13
2	-2	6	-6	10	-10	14	-14
3	-3	7	-7	11	-11	15	-15
-4	4	-8	8	-12	12	-16	16

a) $C(4, 8)$

16	17	12	21	8	25	4	29
18	15	22	11	26	7	30	3
19	14	23	10	27	6	31	2
13	20	9	24	5	28	1	32

b) $R_1(4, 8)$

Figure 2.53: Magic subrectangle of size 4 x 8

Then, all numbers larger than the mean number $\frac{m \cdot n}{2} = 16$ of the previous rectangle are increased by the value $2n = 16$.

16	33	12	37	8	41	4	45
34	15	38	11	42	7	46	3
35	14	39	10	43	6	47	2
13	36	9	40	5	44	1	48

Figure 2.54: $R_1(4, 8)$ with increased numbers

This rectangle now contains the numbers from 1 to 16 and from 33 to 48, leaving a gap of the $2n$ numbers 17 to 32. Therefore, an auxiliary magic rectangle $R_2(2, n)$ is required, whose numbers are increased in each case by $\frac{m \cdot n}{2} = 16$, as it is shown in figure 2.55.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

a) magic rectangle $R_2(2, 8)$

17	31	30	20	21	27	26	24
32	18	19	29	28	22	23	25

b) increased numbers

Figure 2.55: Second subrectangle $R_2(2, 8)$ with increased numbers

Finally, if you insert the two rectangles $R_1(4, 8)$ and $R_2(2, 8)$ one above the other into the target rectangle $R(m, n)$, this is magic as in figure 2.56 . The row sums are 196 and the column sums are 147.

16	33	12	37	8	41	4	45
34	15	38	11	42	7	46	3
35	14	39	10	43	6	47	2
13	36	9	40	5	44	1	48
17	31	30	20	21	27	26	24
32	18	19	29	28	22	23	25

Figure 2.56: Magic rectangle of size 6 x 8 (Bier - Kleinschmidt)

2.6 De Los Reyes - Das - Midha

Another method to create magic rectangles where the number of rows and columns are both even, comes from De Los Reyes, Das and Midha.[7] They used matrices and embed rectangles of smaller size into the target rectangle.

The authors explain the construction method very mathematically with matrices and also use the Kronecker product very often. Since this is difficult to understand for non-mathematicians, I explain the procedure here more simply. However, I leave the names of the involved matrices unchanged for a better comparison.

[7] Reyes - Das - Midha [19]

Let $m = 2p$ be the number of rows and $n = 2q$ the number of columns. First they define some small matrices:

$$Q_e = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \qquad Q_a = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \qquad Q_b = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{pmatrix}$$

In addition, a p x q matrix A with numbers $0, 1, \ldots, pq-1$ is required.

$$A = \begin{pmatrix} 0 & 1 & 2 & \ldots & q-1 \\ q & q+1 & q+2 & \ldots & 2q-1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (p-1)\,q & (p-1)\,q+1 & (p-1)\,q+2 & \ldots & pq-1 \end{pmatrix}$$

So, for a 6 x 8 matrix $p = 3$ and $q = 4$ applies and the corresponding matrix A is

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{pmatrix}$$

This is followed by a 2p x q matrix X, which is generated from the p x q matrix A. The rows of A are taken from top to bottom and transferred to the matrix X. Directly below the transferred row, these numbers are additionally entered in reverse order into an additional row.

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 8 & 9 & 10 & 11 \\ 11 & 10 & 9 & 8 \end{pmatrix}$$

A 2p x 2q matrix Y is then generated from this matrix X. In this step, the numbers of X are each multiplied by 4 and then increased by 1. In addition, this result is also duplicated in the cell to the right. Since this causes each number in the rows to appear twice, the number of columns increases to $2q$.

$$Y = \begin{pmatrix} 1 & 1 & 5 & 5 & 9 & 9 & 13 & 13 \\ 13 & 13 & 9 & 9 & 5 & 5 & 1 & 1 \\ 17 & 17 & 21 & 21 & 25 & 25 & 29 & 29 \\ 29 & 29 & 25 & 25 & 21 & 21 & 17 & 17 \\ 33 & 33 & 37 & 37 & 41 & 41 & 45 & 45 \\ 45 & 45 & 41 & 41 & 37 & 37 & 33 & 33 \end{pmatrix}$$

In the next step, another matrix C is generated from Y, where the left half of the matrix Y is taken over unchanged. In the right half, the numbers of these columns are reversed in pairs, but their relative order to each other is maintained. This means, for example, for the column at the right edge, that the two upper numbers 13 and 1 are entered in exactly this order at the lower edge of this column. At the same time, the numbers 45 and 33 move from the lower to the upper edge. Proceed in the same way with further groups of four numbers, if they exist. If two numbers remain at the end, they are not changed.

$$C = \begin{pmatrix} 1 & 1 & 5 & 5 & 41 & 41 & 45 & 45 \\ 13 & 13 & 9 & 9 & 37 & 37 & 33 & 33 \\ 17 & 17 & 21 & 21 & 25 & 25 & 29 & 29 \\ 29 & 29 & 25 & 25 & 21 & 21 & 17 & 17 \\ 33 & 33 & 37 & 37 & 9 & 9 & 13 & 13 \\ 45 & 45 & 41 & 41 & 5 & 5 & 1 & 1 \end{pmatrix}$$

A matrix B' with two rows is now defined as an auxiliary matrix.

$$B' = \begin{cases} \begin{pmatrix} \underbrace{Q \quad Q \quad \cdots \quad Q}_{\frac{q}{2} \text{ mal}} \end{pmatrix} & \text{if } q \text{ is even} \\ \begin{pmatrix} Q_e \quad Q_a \quad \underbrace{Q_b \quad Q_a}_{\frac{q-3}{3} \text{ mal}} \quad Q_e \end{pmatrix} & \text{if } q \text{ is odd} \end{cases}$$

For $q = 4$ this means that the upper case is true and the matrix Q with $\frac{q}{2} = 2$ is joined exactly twice. Thus, the matrix B' has the following appearance:

$$B' = \begin{pmatrix} \underbrace{\begin{matrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{matrix}}_{Q} & \underbrace{\begin{matrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{matrix}}_{Q} \end{pmatrix}$$

If you now put B' $p = 3$ times one above the other, you get the matrix B which is required.

$$B = \begin{pmatrix} 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \end{pmatrix}$$

Finally, adding component by component the numbers from the matrices B and C, you get the magic rectangle $R(6, 8)$ from figure 2.57 with row sums 196 and column sums 147.

1	4	6	7	41	44	46	47
16	13	11	10	40	37	35	34
17	20	22	23	25	28	30	31
32	29	27	26	24	21	19	18
33	36	38	39	9	12	14	15
48	45	43	42	8	5	3	2

Figure 2.57: Magic rectangle of size 6 x 8 (De Los Reyes - Das - Midha)

Example 8x10

As a second example, the magic rectangle $R(8, 10)$ is to be constructed. With $p = 4$ and $q = 5$ the following auxiliary matrices are obtained.

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 \\ 10 & 11 & 12 & 13 & 14 \\ 14 & 13 & 12 & 11 & 10 \\ 15 & 16 & 17 & 18 & 19 \\ 19 & 18 & 17 & 16 & 15 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 & 5 & 5 & 9 & 9 & 13 & 13 & 17 & 17 \\ 17 & 17 & 13 & 13 & 9 & 9 & 5 & 5 & 1 & 1 \\ 21 & 21 & 25 & 25 & 29 & 29 & 33 & 33 & 37 & 37 \\ 37 & 37 & 33 & 33 & 29 & 29 & 25 & 25 & 21 & 21 \\ 41 & 41 & 45 & 45 & 49 & 49 & 53 & 53 & 57 & 57 \\ 57 & 57 & 53 & 53 & 49 & 49 & 45 & 45 & 41 & 41 \\ 61 & 61 & 65 & 65 & 69 & 69 & 73 & 73 & 77 & 77 \\ 77 & 77 & 73 & 73 & 69 & 69 & 65 & 65 & 61 & 61 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 5 & 5 & 9 & 69 & 73 & 73 & 77 & 77 \\ 17 & 17 & 13 & 13 & 9 & 69 & 65 & 65 & 61 & 61 \\ 21 & 21 & 25 & 25 & 29 & 49 & 53 & 53 & 57 & 57 \\ 37 & 37 & 33 & 33 & 29 & 49 & 45 & 45 & 41 & 41 \\ 41 & 41 & 45 & 45 & 49 & 29 & 33 & 33 & 37 & 37 \\ 57 & 57 & 53 & 53 & 49 & 29 & 25 & 25 & 21 & 21 \\ 61 & 61 & 65 & 65 & 69 & 9 & 13 & 13 & 17 & 17 \\ 77 & 77 & 73 & 73 & 69 & 9 & 5 & 5 & 1 & 1 \end{pmatrix}$$

Since q is odd in this example, the matrix B' is now generated differently.

$$B' = \begin{pmatrix} Q_e & Q_a & \underbrace{Q_b \quad Q_a}_{\frac{q-3}{3}\ \text{mal}} & Q_e \end{pmatrix}$$

With $\frac{q-3}{2} = \frac{5-3}{2} = 1$, the two matrices Q_b and Q_a are thus inserted only once into the matrix Matrix B', which thus looks like

$$B' = \begin{pmatrix} \underbrace{\begin{matrix} 0 & 2 \\ 3 & 1 \end{matrix}}_{Q_e} & \underbrace{\begin{matrix} 2 & 3 \\ 1 & 0 \end{matrix}}_{Q_a} & \underbrace{\begin{matrix} 0 & 1 \\ 3 & 2 \end{matrix}}_{Q_b} & \underbrace{\begin{matrix} 2 & 3 \\ 1 & 0 \end{matrix}}_{Q_a} & \underbrace{\begin{matrix} 0 & 2 \\ 3 & 1 \end{matrix}}_{Q_e} \end{pmatrix}$$

As in the example of the rectangle of size 6 x 8, B' is now inserted p times one above the other into the matrix B.

$$B = \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \end{pmatrix}$$

Finally, the matrices B and C are added again and you get the magic rectangle $R(8, 10)$ from figure 2.58. The row sums are 405 and the column sums are 324.

1	3	7	8	9	70	75	76	77	79
20	18	14	13	12	71	66	65	64	62
21	23	27	28	29	50	55	56	57	59
40	38	34	33	32	51	46	45	44	42
41	43	47	48	49	30	35	36	37	39
60	58	54	53	52	31	26	25	24	22
61	63	67	68	69	10	15	16	17	19
80	78	74	73	72	11	6	5	4	2

Figure 2.58: Magic rectangle of size 8 x 10 (De Los Reyes - Das - Midha)

2.7 Diagonal method

Another way to create magic rectangles is to modify methods for constructing double-even magic squares. Of course, this works only for a few methods, such as the *diagonal method*, which can be transferred easily.[8]

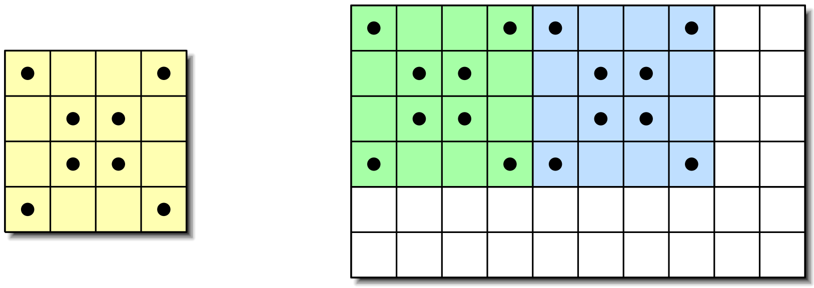
The diagonal method goes back to an unknown author from the Arab region, who wrote a treatise on magic squares in the 12th century.[9] He also presented the diagonal method he had created because he considered it much simpler than other methods known at that time. This method is also described in Moschopoulos' work on magic squares from the Arab region.[10]

First, a base pattern of size 4 x 4 is created, whose diagonals contain markings. If the target rectangle cannot be completely divided into 4 x 4 subsquares, two rows or columns are first separated at the lower or right edge. Then copy the base pattern into all 4 x 4 squares as shown in figure 2.59. The separated rows and columns are ignored at this time.

[8] Danielsson [5] see chapter 5.1.1

[9] Sesiano [21] S. 43 – 45 and Sesiano [20] S. 188-192

[10] Tannery [22]



a) base pattern b) reduced 4 x 8 - rectangle

Figure 2.59: Marked subsquares of size 4 for the target rectangle $R(6, 8)$

The reduced rectangle $R_1(m_1, n_1) = R_1(4, 8)$ is first filled with the numbers from 1 to $m_1 \cdot n_1 = 4 \cdot 8 = 32$ in natural order. Starting with 1 in the upper left corner, the other numbers are written continuously from left to right and from top to bottom in the square. Finally, all numbers z that lie on unmarked cells are replaced by their complement.

$$z \longmapsto m_1 \cdot n_1 + 1 - z \qquad\qquad z \longmapsto 33 - z$$

This creates the magic rectangle of size 4 x 8 from figure 2.60.

1	2	3	4	5	6	7	8		
9	10	11	12	13	14	15	16		
17	18	19	20	21	22	23	24		
25	26	27	28	29	30	31	32		

a) natural order

1	31	30	4	5	27	26	8		
24	10	11	21	20	14	15	17		
16	18	19	13	12	22	23	9		
25	7	6	28	29	3	2	32		

b) complementary numbers

Figure 2.60: Magic rectangle $R_1(4, 8)$

To extend this rectangle by two rows, all numbers larger than the median $\frac{m_1 \cdot n_1}{2} = 16$ of the already existing numbers are increased by the value $2\,n_1 = 16$. This creates a gap of 16 numbers, which is filled with the still missing numbers.

1	47	46	4	5	43	42	8		
40	10	11	37	36	14	15	33		
16	34	35	13	12	38	39	9		
41	7	6	44	45	3	2	48		

Figure 2.61: Rectangle $R_1(4, 8)$ with some numbers increased by 16

So create a rectangle $R_1'(2, n_1)$ and increase all numbers by $2n_1 = 16$.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

a) rectangle $R_1'(2, n_1)$

17	31	30	20	21	27	26	24
32	18	19	29	28	22	23	25

b) increased numbers

Figure 2.62: Magic rectangle $R_1'(2, 8)$

The two rows from the rectangle $R_1'(2, 8)$ can now be added somewhere to the rectangle $R_1(4, 8)$ so that the magic rectangle $R_2(6, 8)$ from figure 2.63 is created.

1	47	46	4	5	43	42	8		
40	10	11	37	36	14	15	33		
16	34	35	13	12	38	39	9		
41	7	6	44	45	3	2	48		
17	31	30	20	21	27	26	24		
32	18	19	29	28	22	23	25		

Figure 2.63: Magic rectangle $R_2(6, 8)$

Now the rectangle only needs to be extended by two columns. Just proceed in the same way as for extending by two rows.

1	59	58	4	5	55	54	8		
52	10	11	49	48	14	15	45		
16	46	47	13	12	50	51	9		
53	7	6	56	57	3	2	60		
17	43	42	20	21	39	38	24		
44	18	19	41	40	22	23	37		

a) $R_2(6, 8)$: some numbers increased

12	1
2	11
10	3
4	9
5	8
6	7

b) $R_2'(6, 2)$

36	25
26	35
34	27
28	33
29	32
30	31

c) increased

Figure 2.64: Subrectangles for horizontal expansion

This expansion creates the magic rectangle $R(6, 10)$ from figure 2.65. The row sums are 305 and the column sums are 183.

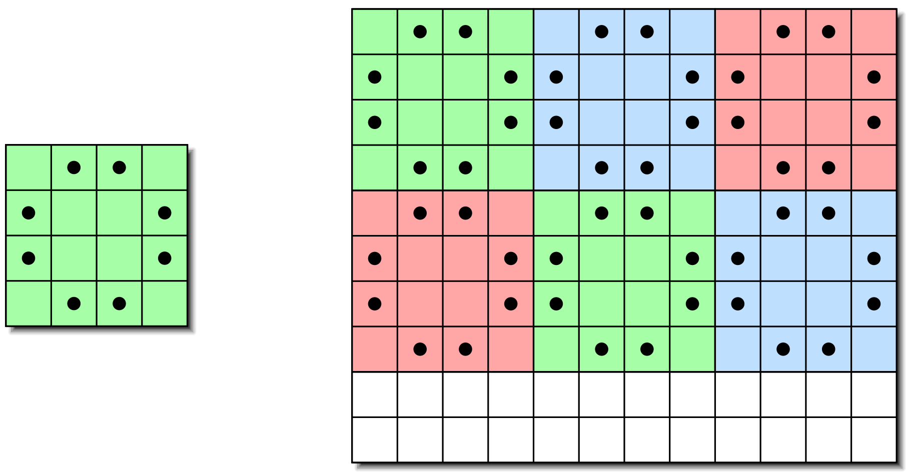
1	59	58	4	5	55	54	8	36	25
52	10	11	49	48	14	15	45	26	35
16	46	47	13	12	50	51	9	34	27
53	7	6	56	57	3	2	60	28	33
17	43	42	20	21	39	38	24	29	32
44	18	19	41	40	22	23	37	30	31

Figure 2.65: Magic rectangle of size 6 x 10 (diagonal method)

Of course, as always with this method, you can permute the rows and columns of all magic subrectangles that appear as intermediate results arbitrarily and independently.

Variants

There are two variants that are used in the following construction of a rectangle of size 10 x 12. First, a different base square can be chosen where the markings are not on the diagonals. With the size chosen in this example, this base pattern is copied into six subsquares.



a) base pattern b) reduced 8 x 12 - rectangle

Figure 2.66: Marked subsquares of size 4 for the target rectangle $R(10, 12)$

As a second variant, numbers in natural order are not assumed, but plus sequences.[11] In this example, step size 2 has been selected, but step size 4 is also always possible.

If you fill the 4 x 4 subsquares with this arrangement and replace the numbers that are not on marked cells with their complementary numbers, you get the rectangle from figure 2.67.

[11] Danielsson [5] see chapter 1.8.2

1	3	2	4	5	7	6	8	9	11	10	12
13	15	14	16	17	19	18	20	21	23	22	24
25	27	26	28	29	31	30	32	33	35	34	36
37	39	38	40	41	43	42	44	45	47	46	48
49	51	50	52	53	55	54	56	57	59	58	60
61	63	62	64	65	67	66	68	69	71	70	72
73	75	74	76	77	79	78	80	81	83	82	84
85	87	86	88	89	91	90	92	93	95	94	96

a) plus sequence with step size 2

96	3	2	93	92	7	6	89	88	11	10	85
13	82	83	16	17	78	79	20	21	74	75	24
25	70	71	28	29	66	67	32	33	62	63	36
60	39	38	57	56	43	42	53	52	47	46	49
48	51	50	45	44	55	54	41	40	59	58	37
61	34	35	64	65	30	31	68	69	26	27	72
73	22	23	76	77	18	19	80	81	14	15	84
12	87	86	9	8	91	90	5	4	95	94	1

b) complementary numbers

Figure 2.67: Magic rectangle $R_1(8, 12)$

In the next step, all numbers that are larger than the center 48 are increased by 24. This again creates space for the 24 numbers to be inserted in the two lower rows.

120	3	2	117	116	7	6	113	112	11	10	109
13	106	107	16	17	102	103	20	21	98	99	24
25	94	95	28	29	90	91	32	33	86	87	36
84	39	38	81	80	43	42	77	76	47	46	73
48	75	74	45	44	79	78	41	40	83	82	37
85	34	35	88	89	30	31	92	93	26	27	96
97	22	23	100	101	18	19	104	105	14	15	108
12	111	110	9	8	115	114	5	4	119	118	1

Figure 2.68: Rectangle $R_1(8, 12)$ with some numbers increased by 24

To extend the rectangle R_1 by two more rows, a rectangle $R'_1(2, 12)$ is needed whose numbers are all increased by 48.

1	23	22	4	5	19	18	8	9	15	14	12
24	2	3	21	20	6	7	17	16	10	11	13

49	71	70	52	53	67	66	56	57	63	62	60
72	50	51	69	68	54	55	65	64	58	59	61

Figure 2.69: Magic rectangle $R'_1(2, 12)$ with increased numbers

The two rows from rectangle $R'_1(2, 12)$ can now be added somewhere to the rectangle $R_1(8, 12)$, so that the magic rectangle $R_2(10, 12)$ from figure 2.70 is created. Since the number of columns is a multiple of 4, this rectangle no longer needs to be extended horizontally. The row sums of this magic rectangle are 726 and the column sums are 605.

120	3	2	117	116	7	6	113	112	11	10	109
13	106	107	16	17	102	103	20	21	98	99	24
25	94	95	28	29	90	91	32	33	86	87	36
84	39	38	81	80	43	42	77	76	47	46	73
48	75	74	45	44	79	78	41	40	83	82	37
85	34	35	88	89	30	31	92	93	26	27	96
97	22	23	100	101	18	19	104	105	14	15	108
12	111	110	9	8	115	114	5	4	119	118	1
49	71	70	52	53	67	66	56	57	63	62	60
72	50	51	69	68	54	55	65	64	58	59	61

Figure 2.70: Magic rectangle of size 10×12 (diagonal method)

3 Rectangles of size 2p+1 x 2q+1

3.1 Planck

C. Planck developed the *method of complementary numbers* to create magic rectangles and squares.[12] For a magic rectangle of size 3 x 5, he arranges the numbers $1, 2, \ldots, 15$ into pairs of equal sum, with no partner number assigned to the number 8.

z_1	z_2	$a = \frac{z_2 - z_1}{2}$
1	15	7
2	14	6
3	13	5
4	12	4
5	11	3
6	10	2
7	9	1
8		

Table 1: Key numbers for a rectangle of size 3 x 5

Each of these pairs of numbers has different differences, which Planck additionally divides by 2. Planck is now looking for two different combinations of numbers like $a = b + c$ among the seven key numbers a in the right column in table 1. Then he inserts their associated numbers into a rectangle S. In addition, the median of all numbers, here the number 8, is always entered in the center of this rectangle.

Altogether, there are exactly three such combinations for the numbers from 1 to 7, but only the upper one from table 2 leads to success.

a	b	c		a	b	c
5	2	3		7	1	6
4	1	3		7	2	5
6	1	5		7	3	4

Table 2: Possible combinations of key numbers

The first equation of this combination is $5 = 2 + 3$. Looking at the two numbers assigned to the key numbers, you can see that the larger number of the pair of numbers assigned to the key number 5 is 13, while the two smaller numbers of the pairs of numbers assigned to the key numbers 2 and 3 are 6 and 5. These three numbers give exactly the required column sum $13 + 6 + 5 = 24$ and are entered in any column of a rectangle S. The corresponding complementary numbers are then entered in the symmetrically located cells.

Figure 3.1: Enter the numbers assigned to the key numbers into rectangle S

[12] Planck [18]

Although both, the column and the order of the three numbers can be chosen arbitrarily, the procedure here is systematic. Therefore, the columns are always filled from left to right, the larger number of the key number a in the upper row and the two smaller numbers below.

The second equation of the chosen combination is $7 = 1 + 6$. This results in the three numbers 15, 7 and 2 for the next column, which also has the column sum 24. These numbers and their corresponding complementary numbers 1, 9 and 14 are again entered in the two columns that are still empty.

a	13	15			
b	6	7	8		
c	5	2			

13	15		14	11
6	7	8	9	10
5	2		1	3

Figure 3.2: Enter the numbers for the second equation

This leaves only the center column. The key number 4 has not been used so far, and the two associated numbers 4 and 12 are entered arbitrarily into the two empty cells of this column. Thus, the rectangle S from figure 3.3 has been created, where all column sums are 24.

13	15	4	14	11
6	7	8	9	10
5	2	12	1	3
24	24	24	24	24

Figure 3.3: Rectangle S with equal column sums

In the next step, a second rectangle Z is constructed, where the numbers are arranged in such a way that all row sums are 40. If there are five numbers in a row, key numbers are now searched with equations that satisfy the condition $d + e = f + g + h$. There are a total of nine combinations of numbers for this condition, which are listed in table 3.

d	e	f	g	h
6	2	4	1	3
7	2	5	1	3
7	3	5	1	4
7	4	6	2	3
7	5	6	2	4
6	4	7	1	2
6	4	5	2	3
6	5	7	1	3
5	4	6	1	2

Table 3: Possible equations $d + e = f + g + h$

However, for all equations

$$a_1 = b_1 + c_1$$
$$a_2 = b_2 + c_2$$
$$d + e = f + g + h$$

two additional conditions exist:

1. the two numbers b_i and c_i, which are on the right side of the equal signs in the first two equations, must not also be on one side in the third equation
2. the numbers a_i and b_i as well as a_i and c_i, which are on different sides, must not also be on different sides in the third equation

For example, choosing $a = b + c$ with numbers $7 = 1 + 6$, the equation $4 + 5 = 1 + 2 + 6$ cannot be used because the numbers 1 and 6 for b and c are both times on the right side. Likewise, the equation $2 + 7 = 1 + 3 + 5$ cannot be used because the numbers 7 and 1 for a and b are both times on different sides. The equation $4 + 7 = 2 + 3 + 6$ is also eliminated, since the numbers 7 and 6 for a and c are on different sides of the equations.

With these two additional conditions, only the upper equation $6 + 2 = 4 + 1 + 3$ from table 3 remain of the original nine equations and lead to a magic rectangle. All other combinations of equations do not satisfy all conditions.

First, the two smaller numbers 2 and 6 of the key numbers 6 and 2 are entered in the bottom row of the two left columns of a rectangle Z. The larger numbers 12, 9 and 11 of the key numbers 4, 1 and 3 are also placed in this row. The order in which the columns are entered is completely arbitrary, as is the selection of the lower or upper row for entering these numbers. Here, however, we proceed systematically from left to right to better compare the results of the different examples. The corresponding complementary numbers are then entered in the symmetrically located cells.

		8		
2	6	12	9	11
d	e	f	g	h

5	7	4	10	14
		8		
2	6	12	9	11

Figure 3.4: Enter the first numbers assigned to the key numbers into rectangle Z

For this rectangle, the key numbers 5 and 7 have not yet been used. The associated numbers 3 and 13 as well as 1 and 15 are now entered in any two horizontally symmetrical pairs of cells.

5	7	4	10	14	40
3	1	8	15	13	40
2	6	12	9	11	40

Figure 3.5: Enter the numbers assigned to the key numbers into rectangle Z

So far, two auxiliary rectangles have been created. The first rectangle S has the column sums 24 and the rectangle Z has the row sums 40 required for the magic rectangle. These two rectangles are finally combined to a magic rectangle by making some shifts in the rectangle Z so that additionally the column sums also match.

Start with the number 5 in the upper left corner of the rectangle Z. At the rectangle S you can see that the numbers 13 and 6 for the column sum 24 are missing. These are in rows 2 and 3 of Z and are moved to the left column to the number 5.

5	7	4	10	14
3	1	8	15	13
2	6	12	9	11

5	7	4	10	14
13	3	1	8	15
6	2	12	9	11
24				

Figure 3.6: Move numbers in Z to the first column

Then the number 7 follows in the upper row of Z. In rectangle S you can see that this number gives the required column sum with corresponding numbers 15 and 2. So, these two numbers are moved to the second column.

5	7	4	10	14
13	3	1	8	15
6	2	12	9	11
24				

5	7	4	10	14
13	15	3	1	8
6	2	12	9	11
24	24			

Figure 3.7: Move numbers in Z to the second column

The next two numbers 4 and 10 in the top row of Z are joined by the numbers 8 and 12 and 3 and 11, which are moved to the appropriate columns accordingly. The row sums are 40 and the column sums are 24.

5	7	4	10	14
13	15	8	3	1
6	2	12	9	11
24	24	24		

5	7	4	10	14
13	15	8	3	1
6	2	12	11	9
24	24	24	24	

Figure 3.8: Move numbers in Z to the third and fourth columns

If now also the last column results in the required column sum, the selected conditions were fulfilled and the magic rectangle from figure 3.9 is created.

5	7	4	10	14	40
13	15	8	3	1	40
6	2	12	11	9	40
24	24	24	24	24	

Figure 3.9: Magic rectangle of size 3 x 5 (Planck)

Example 3x7

For the next example of a 3 x 7 magic rectangle, the initial table with the complementary number pairs is set up again.

z_1	z_2	$a = \frac{z_2 - z_1}{2}$
1	21	10
2	20	9
3	19	8
4	18	7
5	17	6
6	16	5
7	15	4
8	14	3
9	13	2
10	12	1
11		

Table 4: Key numbers for a rectangle of size 3×7

Among the ten key numbers in the right column in table 4, look for three different combinations of numbers like $a = b + c$. With this size, the number 11 is placed in the center of the target rectangle.

With this size, there are already twelve suitable combinations, and all can be used this time.

a	b	c	a	b	c	a	b	c
5	1	4	8	2	6	10	3	7
5	1	4	9	3	6	10	2	8
5	2	3	8	1	7	10	4	6
5	2	3	9	1	8	10	4	6
6	1	5	7	3	4	10	2	8
6	2	4	8	3	5	10	1	9
6	2	4	9	1	8	10	3	7
7	1	6	9	4	5	10	2	8
7	2	5	9	1	8	10	4	6
7	3	4	8	2	6	10	1	9
8	2	6	9	4	5	10	3	7
8	3	5	9	2	7	10	4	6

Table 5: Possible combinations of key numbers

In this example, the following equations have been chosen for the key numbers:

$$10 = 2 + 8 \qquad 7 = 3 + 4 \qquad 6 = 1 + 5$$

As in the first example, the numbers assigned to these key numbers are entered in the left columns of the auxiliary rectangle S. Then the corresponding complementary numbers are placed in the symmetrically located cells.

a	21	18	17				
b	9	8	10	11			
c	3	7	6				

21	18	17		16	15	19
9	8	10	11	12	14	13
3	7	6		5	4	1

Figure 3.10: Enter the numbers assigned to the key numbers into rectangle S

The code 9 still has not been used so far, and the two corresponding numbers 2 and 20 are entered into the still empty cells and all column sums of the rectangle S are 33.

21	18	17	2	16	15	19
9	8	10	11	12	14	13
3	7	6	20	5	4	1
33	33	33	33	33	33	33

Figure 3.11: Rectangle S with equal column sums

The second auxiliary rectangle Z is again constructed in such a way that it has equal row sums. For this size with seven columns, the second condition is now $d + e + f = g + h + i + j$. A total of 171 of these equations exist, but only six of them satisfy the additional conditions for the three equations $a + b = c$. For this construction, the equation

$$2 + 9 + 10 = 3 + 5 + 6 + 7$$

has been chosen. The smaller numbers of the key numbers 2, 9 and 10 are entered into the three left columns of the lower row of Z, followed by the larger numbers of the key numbers 3, 5, 6, and 7. In the next step, the corresponding complementary numbers are placed into the symmetrically located cells.

			11			
9	2	1	14	16	17	18
d	e	f	g	h	i	j

4	5	6	8	21	20	13
			11			
9	2	1	14	16	17	18

Figure 3.12: Enter the numbers assigned to the key numbers into rectangle Z

The key numbers 1, 4, and 8 have not been used in this choice of equation so far, so the associated numbers can be entered in any three horizontally symmetrical pairs of cells.

4	5	6	8	21	20	13	77
10	7	3	11	19	15	12	77
9	2	1	14	16	17	18	77

Figure 3.13: Rectangle Z with equal row sums

Thus, the two required auxiliary rectangles have been created. The first rectangle S has column sums 33 and rectangle Z has row sums 77, which are required for the magic rectangle. This allows the two rectangles to be combined into one magic rectangle.

The numbers in the upper row of rectangle Z remain unchanged and the numbers in the two lower rows are shifted in such a way that the column sums are always 33.

4	5	6	8	21	20	13
15	10	7	3	11	19	12
14	9	2	1	16	17	18
33						

4	5	6	8	21	20	13
15	12	10	7	3	11	19
14	16	9	2	1	17	18
33	33					

4	5	6	8	21	20	13
15	12	10	7	3	11	19
14	16	17	9	2	1	18
33	33	33				

4	5	6	8	21	20	13
15	12	10	7	3	11	19
14	16	17	18	9	2	1
33	33	33	33			

4	5	6	8	21	20	13
15	12	10	7	3	11	19
14	16	17	18	9	2	1
33	33	33	33	33		

4	5	6	8	21	20	13
15	12	10	7	3	11	19
14	16	17	18	9	2	1
33	33	33	33	33	33	

Figure 3.14: Move numbers in the rectangle Z to the appropriate columns

The last columns are already arranged correctly and since the row sums do not change, the magic rectangle shown in figure 3.15 is created. The row sums are 77 and the column sums are 33.

4	5	6	8	21	20	13	77
15	12	10	7	3	11	19	77
14	16	17	18	9	2	1	77
33	33	33	33	33	33	33	

Figure 3.15: Magic rectangle of size 3 x 7 (Planck)

Example 5x7

For a magic rectangle of size 5 x 7, the following initial table results.

z_2	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19	18
z_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
$a = \frac{z_2 - z_1}{2}$	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

Table 6: Key numbers for a rectangle of size 5 x 7

Since there are now five rows and seven columns, the equations needed for the construction of the rectangle S with equal column sums must also be adjusted. This means that three equations $a+b=c+d+e$ are required. For this example, the following equations have been chosen for the five rows.

$$8+9=2+5+10$$
$$12+14=4+7+15$$
$$13+16=1+11+17$$

The numbers belonging to the two key numbers on the left side of the equation are entered into the two top rows. The smaller numbers belonging to the three key numbers on the right side of the equation are entered below them. Since the median 18 is again placed in the center, the following intermediate result is initially obtained.

a	26	30	31				
b	27	32	34				
c	16	14	17	18			
d	13	11	7				
e	8	3	1				

Figure 3.16: Enter the numbers assigned to the key numbers into rectangle S

Then the complementary numbers are placed in the symmetrically located cells.

26	30	31		35	33	28
27	32	34		29	25	23
16	14	17	18	19	22	20
13	11	7		2	4	9
8	3	1		5	6	10

Figure 3.17: Entering the numbers into symmetrically lying cells

The key numbers 3 and 6 have not been used yet, so the associated numbers 15 and 21 as well as 12 and 24 are entered into any two vertically symmetrical pairs of cells. This creates the rectangle S with equal column sums.

26	30	31	15	35	33	28
27	32	34	12	29	25	23
16	14	17	18	19	22	20
13	11	7	24	2	4	9
8	3	1	21	5	6	10

Figure 3.18: Rectangle S with equal column sums

For the second rectangle Z, with seven columns, two equations like

$$f + g + h = k + l + m + n$$

must now be found that also satisfy the additional two conditions. With 35 numbers involved, the number of equations to choose from increases dramatically. In this example, the following two equations are used.

$$f + g + h = k + l + m + n$$
$$2 + 7 + 17 = 4 + 5 + 6 + 11$$
$$8 + 10 + 14 = 1 + 3 + 12 + 16$$

The numbers assigned to the selected key numbers are first entered in the two bottom rows. (see figure 3.19)

			18			
10	8	4	19	21	30	34
16	11	1	22	23	24	29
f	g	h	k	l	m	n

Figure 3.19: Enter the numbers assigned to the key numbers into rectangle Z

Then the complementary partner numbers are placed in the symmetrically located cells. (see figure 3.20)

7	12	13	14	35	25	20
2	6	15	17	32	28	26
			18			
10	8	4	19	21	30	34
16	11	1	22	23	24	29

Figure 3.20: Entering the corresponding complementary numbers

For this rectangle, the key numbers 1 and 4 have not yet been used. The associated numbers are now entered in any two horizontally symmetrical cells. (see figure 3.21)

7	12	13	14	35	25	20
2	6	15	17	32	28	26
9	5	3	18	33	31	27
10	8	4	19	21	30	34
16	11	1	22	23	24	29

Figure 3.21: Entering the numbers not yet used into rectangle Z

So far, two auxiliary rectangles have been created. The first rectangle S has column sums 90 and the rectangle Z has row sums 126, which are required for the magic rectangle. These two rectangles can now combined again to create a magic rectangle by making some shifts in rectangle Z so that the column sums also match. The necessary shifts are shown in figure 3.22.

7	12	13	14	35	25	20
17	2	6	15	32	28	26
31	9	5	3	18	33	27
34	10	8	4	19	21	30
1	16	11	22	23	24	29
90						

7	12	13	14	35	25	20
17	15	2	6	32	28	26
31	18	9	5	3	33	27
34	21	10	8	4	19	30
1	24	16	11	22	23	29
90	90					

7	12	13	14	35	25	20
17	15	26	2	6	32	28
31	18	27	9	5	3	33
34	21	8	10	4	19	30
1	24	16	11	22	23	29
90	90	90				

7	12	13	14	35	25	20
17	15	26	32	2	6	28
31	18	27	3	9	5	33
34	21	8	30	10	4	19
1	24	16	11	22	23	29
90	90	90	90			

7	12	13	14	35	25	20
17	15	26	32	2	6	28
31	18	27	3	5	9	33
34	21	8	30	19	10	4
1	24	16	11	29	22	23
90	90	90	90	90		

7	12	13	14	35	25	20
17	15	26	32	2	6	28
31	18	27	3	5	33	9
34	21	8	30	19	4	10
1	24	16	11	29	22	23
90	90	90	90	90	90	

Figure 3.22: Shift numbers in rectangle Z

With these shifts, the magic rectangle from figure 3.23 is created. The row sums are 126 and the column sums are 90.

7	12	13	14	35	25	20	126
17	15	26	32	2	6	28	126
31	18	27	3	5	33	9	126
34	21	8	30	19	4	10	126
1	24	16	11	29	22	23	126
90	90	90	90	90	90	90	

Figure 3.23: Magic rectangle of size 3 x 19 (Planck)

Example 3x7

To understand Planck's method even better, a second example for a magic rectangle of size 3 x 7 should be given. With equations

$$6 = 2 + 4 \qquad 2 + 7 + 10 = 1 + 4 + 5 + 9$$
$$9 = 1 + 8$$
$$10 = 3 + 7$$

the magic rectangle from figure 3.24 is created from rectangles S and Z.

17	20	21	6	18	19	15
9	10	8	11	14	12	13
7	3	4	16	1	2	5

a) S

2	6	7	10	21	18	13
19	11	17	3	8	14	5
12	16	9	20	4	1	15

b) Z

2	6	7	10	21	18	13
19	11	17	3	8	14	5
12	16	9	20	4	1	15

c) magic rectangle

Figure 3.24: Magic rectangle of size 3 x 7 (Planck)

Example 5x7

Likewise, a second example for a rectangle of the size 5 x 7 shall be given, where the suitable equations are much more difficult to find because of the large number of theoretical possibilities. With equations

$$\begin{aligned} 1 + 10 &= 2 + 4 + 5 \\ 7 + 17 &= 3 + 8 + 13 \\ 14 + 16 &= 6 + 9 + 15 \end{aligned} \qquad \begin{aligned} 7 + 8 + 16 &= 1 + 4 + 12 + 14 \\ 9 + 11 + 13 &= 3 + 5 + 10 + 15 \end{aligned}$$

the magic rectangle from figure 3.25 results.

19	25	32	7	33	31	23
28	35	34	6	27	26	22
16	15	12	18	24	21	20
14	10	9	30	2	1	8
13	5	3	29	4	11	17

a) S

4	6	14	17	34	26	25
3	8	13	15	31	29	27
16	12	1	18	35	24	20
9	7	5	21	23	28	33
11	10	2	19	22	30	32

b) Z

4	6	14	17	34	26	25
27	29	13	8	3	31	15
24	18	16	20	12	1	35
33	7	28	23	9	21	5
2	30	19	22	32	11	10

c) magic rectangle

Figure 3.25: Magic rectangle of size 5 x 7 (Planck)

3.2 Hagedorn

Rectangles 3xn

Thomas R. Hagedorn has presented a method to generate magic rectangles $R(m, n)$ when m and n are odd.[13] He always starts with rectangles of size 3 x n, whose construction is therefore presented first. For clarity, many rectangles are given here as matrices, as in the original article.

In principle, two cases $n \equiv 1 \mod 4$ and $n \equiv 3 \mod 4$ must be distinguished, in which different procedures are used. In both cases, two matrices $B_+(i)$ and $B_-(i)$ are required, with which an already existing smaller rectangle is filled to form the desired target rectangle.[14]

$$B_+(i) = \begin{pmatrix} i+1 & n+1-i \\ \frac{3n+1}{2}+i & \frac{3n+1}{2}-i \\ 3n-2i & 2n+2i \end{pmatrix} \qquad B_-(i) = \begin{pmatrix} 3n-2i & 2n+2i \\ \frac{3n+1}{2}+i & \frac{3n+1}{2}-i \\ i+1 & n+1-i \end{pmatrix}$$

[13] Hagedorn [8]

[14] As in the original article, the rectangles are partially represented as matrices to ensure better comparability

For both matrices, note that all column sums are $\frac{3\,(3n+1)}{2}$. The row sums of $B_+(i)$ are $n+2$, $3n+1$ and $5n$ and those of $B_-(i)$ are $5n$, $3n+1$ And $n+2$. So if you combine these two matrices, the resulting matrix has the same row and column sums.

Example 3x9

In the first example, a rectangle $R(3,9)$ is created, which is assigned to the case $n \equiv 1 \mod 4$. In this case, you start with the matrix A_1.

$$A_1 = \begin{pmatrix} 1 & 2n & \frac{n+3}{2} \\ 3n & \frac{n+1}{2} & n+1 \\ \frac{3n+1}{2} & 2n+1 & 3n-1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 1 & 18 & 6 \\ 27 & 5 & 10 \\ 14 & 19 & 26 \end{pmatrix}$$

The matrix A_1 is supplemented by $\frac{n-5}{4}$ matrices of the form $B_+(i)$, where i takes the values $1, 2, \ldots, \frac{n-5}{4}$. Additionally, $\frac{n-1}{4}$ matrices $B_-(i)$ are added for $i = \frac{n-1}{4}, \ldots, \frac{n-3}{2}$. In the example for $n = 9$ with $\frac{n-5}{4} = 1$, $\frac{n-1}{4} = 2$ and $\frac{n-3}{2} = 3$, thus one matrix $B_+(i)$ and two matrices $B_-(i)$ are added.

$$B_+(1) = \begin{pmatrix} 2 & 9 \\ 15 & 13 \\ 25 & 20 \end{pmatrix} \qquad B_-(2) = \begin{pmatrix} 23 & 22 \\ 16 & 12 \\ 3 & 8 \end{pmatrix} \qquad B_-(3) = \begin{pmatrix} 21 & 24 \\ 17 & 11 \\ 4 & 7 \end{pmatrix}$$

With these matrices, the magic rectangle $R(3,9)$ from figure 3.26 is created.

1	18	6	2	9	23	22	21	24
27	5	10	15	13	16	12	17	11
14	19	26	25	20	3	8	4	7

A_1 $B_+(1)$ $B_-(2)$ $B_-(3)$

Figure 3.26: Magic rectangle of size 3 x 9 (Hagedorn)

Example 3x11

The second case $n \equiv 3 \mod 4$ shall be presented at a rectangle with $n = 11$. Here, you start with the matrix A_3.

$$A_3 = \begin{pmatrix} 1 & \frac{n+3}{2} \\ 3n & n+1 \\ \frac{3n+1}{2} & 3n-1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 7 \\ 33 & 12 \\ 17 & 32 \end{pmatrix}$$

For another auxiliary matrix B a constant $c = \left\lfloor \frac{n+1}{3} \right\rfloor$ is required. Thus, the difference $n - 3c$ always takes one of the three values 0 or ± 1, as shown in table 7 for some values of n.

n	$\left\lfloor \frac{n+1}{3} \right\rfloor$	c	$n-3c$
7	$\left\lfloor \frac{8}{3} \right\rfloor$	2	1
11	$\left\lfloor \frac{12}{3} \right\rfloor$	4	-1
15	$\left\lfloor \frac{16}{3} \right\rfloor$	5	0
19	$\left\lfloor \frac{20}{3} \right\rfloor$	6	1
23	$\left\lfloor \frac{24}{3} \right\rfloor$	8	-1
27	$\left\lfloor \frac{28}{3} \right\rfloor$	9	0

Table 7: Values of the constant c for some values n

With this constant c the matrix B is defined.

$$B = \begin{cases} \begin{pmatrix} c+1 & 2n+1 & 2n+2c \\ \frac{3n+1}{2}+c & \frac{n+1}{2} & \frac{3n+1}{2}-c \\ 3n-2c & 2n & n+1-c \end{pmatrix} & \text{for } n-3c=1 \\ \begin{pmatrix} c+1 & 2n & 2n+2c \\ \frac{3n+1}{2}+c & \frac{n+1}{2} & \frac{3n+1}{2}-c \\ 3n-2c & 2n+1 & n+1-c \end{pmatrix} & \text{for } n-3c=0 \\ \begin{pmatrix} 3n-2c & 2n+1 & n+1-c \\ \frac{3n+1}{2}+c & \frac{n+1}{2} & \frac{3n+1}{2}-c \\ c+1 & 2n & 2n+2c \end{pmatrix} & \text{for } n-3c=-1 \end{cases}$$

To construct the magic rectangle, start with the three auxiliary matrices A, B and $B_{-}(1)$.

$$A_3 = \begin{pmatrix} 1 & 7 \\ 33 & 12 \\ 17 & 32 \end{pmatrix} \qquad B = \begin{pmatrix} 25 & 23 & 8 \\ 21 & 6 & 13 \\ 5 & 22 & 30 \end{pmatrix} \qquad B_{-}(1) = \begin{pmatrix} 31 & 24 \\ 18 & 16 \\ 2 & 11 \end{pmatrix}$$

Putting these three matrices together gives the rectangle of size 3 x 7 from figure 3.28, where all row and column sums are equal to 119 and 51, respectively.

1	7	25	23	8	31	24				
33	12	21	6	13	18	16				
17	32	5	22	30	2	11				

A_3 $\qquad$ B $\qquad$ $B_{-}(1)$

Figure 3.27: Auxiliary rectangle of size 3 x 7

For the rectangle $R(3, 11)$, you have to add $\frac{n-7}{4}$ matrices $B_+(i)$ for $2 \le i \le \frac{n-3}{4}$ and the same number of matrices $B_-(i)$ for $\frac{n+1}{4} \le i \le \frac{n-3}{2}$, but excluding $i = c$.

By adding these matrices, the row sums do not change anymore and the magic rectangle $R(3, n)$ is created. So, for the chosen example with $n = 11$, only one matrix $B_+(2)$ and also one matrix $B_-(3)$ have to be added because the value of $i = c = 4$ is skipped. The magic rectangle $R(3, 11)$ created in this way is shown in figure 3.28. It has row sums 187 and column sums 51.

1	7	25	23	8	31	24	3	10	27	28
33	12	21	6	13	18	16	19	15	20	14
17	32	5	22	30	2	11	29	26	4	9

$B_+(2)$ $B_-(3)$

Figure 3.28: Magic rectangle of size 3 x 11 (Hagedorn)

Example 5x9

If a magic rectangle with more than three rows is to be constructed, the rectangle $R(m-2, n)$ is extended step by step to the rectangle $R(m, n)$ by adding two rows. Thus, for the rectangle $R(5, 9)$, a rectangle $R(3, 9)$ must be created first.

1	18	6	2	9	23	22	21	24
27	5	10	15	13	16	12	17	11
14	19	26	25	20	3	8	4	7

Figure 3.29: Magic rectangle $R(3, 9)$

In general, the initial rectangle $R(m-2, n)$ contains the numbers from 1 to $mn - 2n$. To all numbers of this rectangle, n is added, so that the numbers from $n+1$ to $mn - n$ are included after the addition. With $d = \frac{m-1}{2}$ one of these rows contains the numbers from d_1 to d_2 except for the number $x = dn + \frac{n+1}{2}$.

$$d_1 = dn + d - 1 \qquad \text{to} \qquad d_2 = dn + n + 1 - d$$

In this example for a rectangle with $m = 5$ rows, the constant $n = 9$ is first added and the rectangle $R'(3, 9)$ is obtained.

10	27	15	11	18	32	31	30	33
36	14	19	24	22	25	21	26	20
23	28	35	34	29	12	17	13	16

Figure 3.30: Rectangle $R'(3, 9)$ with increased numbers

With $d = \frac{m-1}{2} = \frac{5-1}{2} = 2$ you get the required constants.

$$d = 2 \qquad d_1 = d \cdot n + d - 1 = 19 \qquad d_2 = d \cdot n + n + 1 - d = 26 \qquad x = dn + \frac{n+1}{2} = 23$$

Now two columns are searched in which the two numbers d_1 and d_2 occur. These two columns are moved to the left edge of the rectangle, for example, by exchanging with the columns existing there.

10	27	15	11	18	32	31	30	33
36	14	19	24	22	25	21	26	20
23	28	35	34	29	12	17	13	16

15	30	10	11	18	32	31	27	33
19	26	36	24	22	25	21	14	20
35	13	23	34	29	12	17	28	16

Figure 3.31: Shift two columns to the left margin

Now there are $\frac{n-5}{2}$ numbers which are smaller than $\frac{n}{2}$ and different from d and $d-1$, respectively. For $n = 9$, these are the numbers 3 and 4. These numbers are divided into two groups a_i and b_i, distinguishing two cases.

If $n \equiv 1 \mod 4$ applies, $\frac{n-1}{4}$ numbers are assigned to the group a_i and $\frac{n-9}{4}$ numbers to group b_i. In the other case $n \equiv 3 \mod 4$, these numbers are divided so that the two groups contain $\frac{n-3}{4}$ and $\frac{n-7}{4}$ numbers, respectively. For the example given, the first case applies and with $\frac{n-1}{4} = \frac{9-1}{4} = 2$, the two numbers 3 and 4 are both assigned to the group a_i, while b_i remains empty.

To construct the magic rectangle with this number of columns, further matrices are now required.

$$A = \begin{pmatrix} d-1 & d & n+2-d & n+1-d \\ mn+2-d & mn+1-d & mn-n+d-1 & mn-n+d \end{pmatrix}$$

$$C_+(i) = \begin{pmatrix} i & n+1-i \\ mn+1-i & mn-n+i \end{pmatrix} \qquad C_-(i) = \begin{pmatrix} mn+1-i & mn-n+i \\ i & n+1-i \end{pmatrix}$$

$$D_n = \begin{cases} \begin{pmatrix} \frac{n+1}{2} \\ mn + \frac{1-n}{2} \end{pmatrix} & \text{for } n \equiv 1 \mod 4 \\ \begin{pmatrix} mn + \frac{1-n}{2} \\ \frac{n+1}{2}] \end{pmatrix} & \text{for } n \equiv 3 \mod 4 \end{cases}$$

First, the matrix A is inserted into an auxiliary rectangle $H(2,n)$ with only two rows. Then for each number from a_i the matrices $C_-(a_i)$ and for each number from b_i the matrices $C_+(b_i)$ follow, before the matrix D_n forms the end.

For the current example, this means first the matrix A, then for the numbers 3 and 4 the two matrices $C_-(3)$ and $C_-(4)$. Since the group bi is empty, no matrix $C_+(b_i)$ will be inserted and D_9 forms the end. With matrices

$$A = \begin{pmatrix} 1 & 2 & 9 & 8 \\ 45 & 44 & 37 & 38 \end{pmatrix} \qquad C_-(3) = \begin{pmatrix} 43 & 39 \\ 3 & 7 \end{pmatrix} \qquad C_-(4) = \begin{pmatrix} 42 & 40 \\ 4 & 6 \end{pmatrix} \qquad D_9 = \begin{pmatrix} 5 \\ 41 \end{pmatrix}$$

the auxiliary rectangle $H(2,9)$ in figure 3.32 is determined.

1	2	9	8	43	39	42	40	5
45	44	37	38	3	7	4	6	41

$\underbrace{\qquad\qquad}_{A} \quad \underbrace{\quad}_{C_-(3)} \quad \underbrace{\quad}_{C_-(4)} \quad \underbrace{\,}_{D_n}$

Figure 3.32: Auxiliary rectangle $H(2,9)$

Now all that remains is to join the two rectangles $R(3,9)$ and $H(2,9)$ from figures 3.31 and 3.32 together to create the rectangle of size 5 x 9 from figure 3.33.

15	30	10	11	18	32	31	27	33
19	26	36	24	22	25	21	14	20
35	13	23	34	29	12	17	28	16
1	2	9	8	43	39	42	40	5
45	44	37	38	3	7	4	6	41

Figure 3.33: Filled rectangle of size 5 x 9

Due to the selected auxiliary rectangles, all column sums are already all the same with 115. However, the two lower rows with sums 225 and 189 do not yet have the required row sum 207.

However, this problem can be solved by swapping the numbers d_1 and d_2 as well as d_3 and d_4.

$$d_1 = d - 1 \qquad d_2 = dn + d - 1 \qquad d_3 = mn + 1 - d \qquad d_4 = dn + n + 1 - d$$

In this example, $d_1 = 1$ and $d_2 = 19$ applies, which are in column 1, and $d_3 = 44$ and $d_4 = 26$ in column 2. These swaps result in the magic rectangle $R(5,9)$, shown in figure 3.34. The row sums are 207 and the column sums 115.

15	30	10	11	18	32	31	27	33
1	44	36	24	22	25	21	14	20
35	13	23	34	29	12	17	28	16
19	2	9	8	43	39	42	40	5
45	26	37	38	3	7	4	6	41

Figure 3.34: Magic rectangle of size 5 x 9 (Hagedorn)

Another example for the case $n \equiv 3 \mod 4$ is given with the rectangle $R(5, 11)$. First, you get the rectangle $R'(3, 11)$, where all numbers are increased by 11.

1	7	25	23	8	31	24	3	10	27	28
33	12	21	6	13	18	16	19	15	20	14
17	32	5	22	30	2	11	29	26	4	9

12	18	36	34	19	42	35	14	21	38	39
44	23	32	17	24	29	27	30	26	31	25
28	43	16	33	41	13	22	40	37	15	20

Figure 3.35: Auxiliary rectangles $R(3, 11)$ and $R'(3, 11)$

With $d = 2$, you first determine the columns to be shifted, in which the two numbers $d_1 = d \cdot n + d - 1 = 23$ and $d_2 = d \cdot n + n + 1 - d = 32$ occur. Here the columns 2 and 3 are found, and these two columns are shifted to the left margin.

12	18	36	34	19	42	35	14	21	38	39
44	23	32	17	24	29	27	30	26	31	25
28	43	16	33	41	13	22	40	37	15	20

18	36	12	34	19	42	35	14	21	38	39
23	32	44	17	24	29	27	30	26	31	25
43	16	28	33	41	13	22	40	37	15	20

Figure 3.36: Shift two columns to the left margin

With $\frac{n-5}{2} = \frac{11-5}{2} = 3$ there are now three numbers 3, 4 and 5 which are smaller than $\frac{n}{2}$. Now, the numbers 3 and 4 form the group a_i, while group b_i contains the number 5.

For the auxiliary rectangle H, the rectangles $C_-(3)$ and $C_-(4)$ are now required in addition to the rectangle A for the two numbers from a_i. Since b_i is not empty in this example, but contains the number 5, the rectangle $C_+(5)$ is also inserted. D_{11} then forms the end again. If all auxiliary rectangles are inserted into H, the rectangle from figure 3.37 is created.

1	2	11	10	53	47	52	48	5	7	50
55	54	45	46	3	9	4	8	51	49	6
A				$C_-(3)$		$C_-(4)$		$C_+(5)$		D_{11}

Figure 3.37: Auxiliary rectangle $H(2, 11)$

If you insert the two rectangles $H(2, 11)$ and $R'(3, 11)$ into the target rectangle, you get the rectangle from figure 3.38, where again the column sums are equal to 140. But again the two lower rows with 330 and 286 deviate from the required row sum 308.

18	36	12	34	19	42	35	14	21	38	39
23	32	44	17	24	29	27	30	26	31	25
43	16	28	33	41	13	22	40	37	15	20
1	2	11	10	53	47	52	48	5	7	50
55	54	45	46	3	9	4	8	51	49	6

Figure 3.38: Filled rectangle of size 5 x 11

In the next step, the columns of the numbers $d_1 = 1$, $d_2 = 23$, $d_3 = 54$ And $d_4 = 32$ are searched. The exchange of d_1 and d_2 as well as d_3 and d_4 then creates the magic rectangle from figure 3.39 with row sums 308 and column sums 140.

18	36	12	34	19	42	35	14	21	38	39
1	54	44	17	24	29	27	30	26	31	25
43	16	28	33	41	13	22	40	37	15	20
23	2	11	10	53	47	52	48	5	7	50
55	32	45	46	3	9	4	8	51	49	6

Figure 3.39: Magic rectangle of size 5 x 11 (Hagedorn)

3.3 Bier - Rogers

Thomas Bier and Douglas G. Rogers have presented a method to generate magic rectangles $R(3, n)$ when the number of columns is odd.[15] They distinguish three cases for different values of n.

They denote the upper left corner of the coordinate system as origin $(1, 1)$. The rows i are thus labeled from top to bottom with $1, 2, 3, \ldots$ and the columns j from left to right with $1, 2, 3, \ldots$.

Case 1: n=6m-1

To simplify the representation, a constant $c = j \bmod 6$ is defined. For columns $1 \leq j \leq 6m - 6$ and $m > 1$ the numbers $r_{i,j}$ of the rectangle are calculated using the expressions from table 8.

$r_{1,j} = j$	$r_{2,j} = 9m - \frac{j+1}{2}$	$r_{3,j} = 18m - 2 - \frac{j+1}{2}$	for $c \equiv 1$
$r_{1,j} = 15m - 2 - \frac{j}{2}$	$r_{2,j} = j$	$r_{3,j} = 12m - 1 - \frac{j}{2}$	for $c \equiv 2$
$r_{1,j} = 9m - \frac{j+1}{2}$	$r_{2,j} = 18m - 2 - \frac{j+1}{2}$	$r_{3,j} = j$	for $c \equiv 3$
$r_{1,j} = 12m - 1 - \frac{j}{2}$	$r_{2,j} = 15m - 2 - \frac{j}{2}$	$r_{3,j} = j$	for $c \equiv 4$
$r_{1,j} = 18m - 2 - \frac{j+1}{2}$	$r_{2,j} = j$	$r_{3,j} = 9m - \frac{j+1}{2}$	for $c \equiv 5$
$r_{1,j} = j$	$r_{2,j} = 12m - 1 - \frac{j}{2}$	$r_{3,j} = 15m - 2 - \frac{j}{2}$	for $c \equiv 0$

Table 8: Expressions for columns $1 \leq j \leq 6m - 6$

In the range of columns $6m - 5 \leq j \leq 6m - 1$, on the other hand, the expressions from table 9 are used.

[15] Bier - Rogers [3]

$r_{1,j} = 9\,m - \dfrac{j+1}{2}$	$r_{2,j} = 18\,m - 2 - \dfrac{j+1}{2}$	$r_{3,j} = j$	for $j = 6m - 5$
$r_{1,j} = 15\,m - 2 - \dfrac{j}{2}$	$r_{2,j} = j$	$r_{3,j} = 12\,m - 1 - \dfrac{j}{2}$	for $j = 6m - 4$
$r_{1,j} = j$	$r_{2,j} = 9\,m - \dfrac{j+1}{2}$	$r_{3,j} = 18\,m - 2 - \dfrac{j+1}{2}$	for $j = 6m - 3$
$r_{1,j} = j$	$r_{2,j} = 15\,m - 2 - \dfrac{j}{2}$	$r_{3,j} = 12\,m - 1 - \dfrac{j}{2}$	for $j = 6m - 2$
$r_{1,j} = 18\,m - 2 - \dfrac{j+1}{2}$	$r_{2,j} = j$	$r_{3,j} = 9\,m - \dfrac{j+1}{2}$	for $j = 6m - 1$

Table 9: Expressions for columns $6m - 5 \leq j \leq 6m - 1$

For example, if the rectangle has $n = 6m - 1 = 5$ columns, $m = 1$ applies. This results in the magic rectangle from figure 3.40, where only expressions from table 9 are used.

8	12	3	4	13
15	2	7	11	5
1	10	14	9	6

Figure 3.40: Magic rectangle of size 3 x 5 (Bier - Rogers)

For $n = 11$, $m = 2$ applies and all expressions are now used. The numbers for columns $1 \leq j \leq 6m-6 = 6$ are calculated using table 8, and the numbers for columns $6m - 5 \leq j \leq 6m - 1$, i.e. $7 \leq j \leq 11$, with the expressions from table 9. The resulting magic rectangle $R(3, 11)$ is shown in figure 3.41.

1	27	16	21	31	6	14	24	9	10	28
17	2	32	26	5	20	30	8	13	23	11
33	22	3	4	15	25	7	19	29	18	12

Figure 3.41: Magic rectangle of size 3 x 11 (Bier - Rogers)

The next larger magic rectangle generated with these expressions is the rectangle $R(3, 17)$ with $m = 3$. This rectangle is shown in figure 3.42.

1	42	25	33	49	6	7	39	22	30	46	12	20	36	15	16	43
26	2	50	41	5	32	23	8	47	38	11	29	45	14	19	35	17
51	34	3	4	24	40	48	31	9	10	21	37	13	28	44	27	18

Figure 3.42: Magic rectangle of size 3 x 17 (Bier - Rogers)

Case 2: n=6m+1

For this number of columns, four different tables must be used to calculate the numbers.

Case 2.1: $1 \leq j \leq 3\,m-2$

The expressions from table 10 are used for these columns.

$r_{i,j} = j$	$r_{i+1,j} = 12\,m + 4 - 2\,j$	$r_{i+2,j} = 15\,m + 2 + j$

Table 10: Expressions for columns $1 \leq j \leq 3\,m-2$

However, in this case one special feature has to be considered because the row numbers are not fixed defined. The column numbers are always cyclically assigned to row numbers 1, 2, 3 from left to right. So, column numbers 1, 2, 3 are assigned to row numbers 1, 2, 3, likewise again for column numbers 4, 5, 6 and so on. This assigned row number indicates the start row i, in which the first calculated number $r_{i,j}$ of this column is entered. The two further numbers of this column are then entered one row below from a cyclical perspective.

In the example for $n = 6 \cdot 2 + 1 = 13$ columns, this case occurs for columns 1 to 4. The column $j = 1$ is assigned to initial row 1, so that the first of the three numbers is placed into row 1 and the other two numbers 26 and 33 are entered in the two rows below. Column 2 defines the initial row 2, so that the first number 2 is entered in the center row and the other two, seen cyclically, below it. Column 3 means that the initial row 3, so that the three numbers 3, 22 and 35 of this column are entered in the rows 3, 1 and 2. And column 4 is again assigned to initial row 1, so that the numbers 4, 20 and 36 are entered in rows 1, 2 and 3 again.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	34	22	4									
2	26	2	35	20									
3	33	24	3	36									

Case 2.2: $3m-1 \leq j \leq 3m+2$

The numbers in these columns can be calculated directly with the following expressions.

$r_{1,j} = 6\,m + 6$	$r_{2,j} = 3\,m - 1$	$r_{3,j} = 18\,m + 1$	for $j = 3m - 1$
$r_{1,j} = 18\,m + 2$	$r_{2,j} = 6\,m + 4$	$r_{3,j} = 3\,m$	for $j = 3m$
$r_{1,j} = 6\,m + 2$	$r_{2,j} = 18\,m + 3$	$r_{3,j} = 3\,m + 1$	for $j = m + 1$
$r_{1,j} = 12\,m + 1$	$r_{2,j} = 3\,m + 2$	$r_{3,j} = 12\,m + 3$	for $j = m + 2$

Table 11: Expressions for columns $3m-1 \leq j \leq 3m+2$

Case 2.3: $3\,m+3 \leq j \leq 6\,m-1$

The expressions for calculating the numbers for these columns are

$r_{i,j} = 9\,m + 1 + j$	$r_{i+1,j} = 18\,m + 5 - 2\,j$	$r_{i+2,j} = j$

Table 12: Expressions for columns $3\,m+3 \leq j \leq 6\,m-1$

However, the associated rows are not immediately apparent. The column numbers are calculated again modulo 3. If the result is 0, the three rows of this column are filled in the order 1, 2, 3. If the result is 1, the order is 3, 1, 2, otherwise 2, 3, 1. Again, only the starting row i of the first number is important because the two other numbers of this column are again cyclically entered one row below. [16]

This special case occurs with 13 columns for columns 9, 10 and 11. For column 9, $10 \mod 3 \equiv 0$ applies, so that the three numbers 28, 23 and 9 are entered in rows 1, 2 and 3. For column 10, on the other hand, $10 \mod 3 \equiv 1$ applies, so that the numbers 29, 21 and 10 must be entered in rows 3, 1 and 2. Correspondingly, the numbers in column 11 are placed in rows 2, 3 and 1.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1									28	21	11		
2									23	10	30		
3									9	29	19		

Case 2.4: $6m \leq j \leq 6m + 1$

This case only concerns the last two columns and no special features have to be considered, but the numbers to be entered can be calculated directly.

$r_{1,j} = 6m$	$r_{2,j} = 15m + 1$	$r_{3,j} = 6m + 5$	for $j = 6m$
$r_{1,j} = 15m + 2$	$r_{2,j} = 6m + 3$	$r_{3,j} = 6m + 1$	for $j = 6m + 1$

Table 13: Expressions for columns $6m \leq j \leq 6m + 1$

Altogether, the magic rectangle from figure 3.43 results. In this rectangle, the columns for the four cases are specially marked.

1	34	22	4	18	38	14	25	28	21	11	12	32
26	2	35	20	5	16	39	8	23	10	30	31	15
33	24	3	36	37	6	7	27	9	29	19	17	13

Figure 3.43: Magic rectangle of size 3 x 13 (Bier - Rogers)

The next larger magic rectangle generated with these expressions is the rectangle $R(3, 19)$ with $m = 3$. This rectangle is shown in figure 3.44.

1	49	34	4	52	28	7	24	56	20	37	40	33	14	43	27	17	18	47
38	2	50	32	5	53	26	8	22	57	11	35	13	42	29	16	45	46	21
48	36	3	51	30	6	54	55	9	10	39	12	41	31	15	44	25	23	19

Figure 3.44: Magic rectangle of size 3 x 19 (Bier - Rogers)

[16] the expressions have been rearranged compared to the original article so that the numbers can also be displayed within a column here

Case 3: n is a multiple of 3

If the number of columns for a magic rectangle $R(m,n)$ with $n = 3s$ is a multiple of 3, the rectangle must be generated completely differently. Bier and Rogers assume certain balanced magic rectangles $R(m,s)$, which can be extended to the desired rectangle.

Start with a rectangle $R(3,9)$, i.e. $n = 3s = 3 \cdot 3 = 9$, which can be used to illustrate the process of duplication particularly well because of the manageable size.

You need a magic initial rectangle $R_0(3,3)$ and create two more rectangles, where the numbers are increased by $3s = 9$ and $6s = 18$, respectively, as shown in figure 3.45.

1	6	8
9	2	4
5	7	3

10	15	17
18	11	13
14	16	12

19	24	26
27	20	22
23	25	21

Figure 3.45: $R_0(3,3)$ and the rectangles with increased numbers

A new rectangle is then formed from the nine numbers of the corresponding columns of these rectangles by entering the three columns from left to right. The left column of each of the three rectangles in figure 3.45 thus results in the rectangle S_1 in figure 3.46.

1	6	8
9	2	4
5	7	3

10	15	17
18	11	13
14	16	12

19	24	26
27	20	22
23	25	21

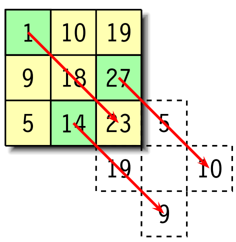
1	10	19
9	18	27
5	14	23

Figure 3.46: Construction of rectangle S_1

The numbers of rectangle S_1 are then transferred to another rectangle R_1. Start in the upper left corner and take the three numbers 1, 18 and 23 of rectangle S_1 diagonally to the lower right. These numbers form the upper row of the rectangle R_1.

The new starting point of the next three numbers is obtained by cyclically moving one row down and one column to the left from the last starting point with number 1 in S_1. In this example, the second starting point is 27 and the next three numbers 27, 5 and 10 can be read diagonally down to the right and entered into the second row of R_1.

The third starting point of the remaining number sequence is 14 and the numbers 14, 19 and 9 can be transferred as in figure 3.47.



a) S_1

1	18	23
27	5	10
14	19	9

b) R_1

Figure 3.47: Transferring the numbers from S_1 to R_1

This auxiliary rectangle R_1 constructed in this way has the special property that all row and column sums are equal. Now it can be transferred to one of the column positions 1, $s+1$ or $2s+1$ in the target rectangle, as it is shown in figure 3.49.

Proceed in the same way with columns 2 and 3 of the rectangles from figure 3.45 and first create the rectangles S_2 and S_3 by reading out the corresponding columns. From these, the numbers are read diagonally again and the corresponding rectangles R_2 and R_3 are formed, which all have the same row and column sums.

6	15	24
2	11	20
7	16	25

a) S_2

8	17	26
4	13	22
3	12	21

b) S_3

6	11	25
20	7	15
16	24	2

c) R_2

8	13	21
22	3	17
12	26	4

d) R_3

Figure 3.48: Construction of rectangles R_2 and R_3

If you also insert these two rectangles into the target rectangle, you will get the magic rectangle from figure 3.49.

1	18	23	6	11	25	8	13	21
27	5	10	20	7	15	22	3	17
14	19	9	16	24	2	12	26	4

R_1 R_2 R_3

Figure 3.49: Magic rectangle of size 3 x 9 (Bier - Rogers)

As you can clearly see, the original rectangle R_0 has been duplicated in a special way. The row sums are $3 \cdot 42 = 126$ and the column sums are 42.

Example 3x15

The rectangle $R(3, 15)$ in the next example has $3s = 3 \cdot 5 = 15$ columns. For the initial rectangle $R_0(3, 5)$, I didn't follow the complex proposal of Bier and Rogers, but use an arbitrary magic rectangle of this size, which can be generated for example with the procedure described in this section for $n = 6m - 1$.

The rectangle from figure 3.40 is used as the initial rectangle $R_0(3, 5)$ from page 51, and the numbers of this rectangle are increased by $3s = 15$ and $6s = 30$ for the two other rectangles, respectively.

8	12	3	4	13
15	2	7	11	5
1	10	14	9	6

23	27	18	19	28
30	17	22	26	20
16	25	29	24	21

38	42	33	34	43
45	32	37	41	35
31	40	44	39	36

Figure 3.50: $R_0(3, 5)$ and the rectangles with increased numbers

Now proceed as in the previous example. The nine numbers from the left columns provide the first rectangle S_1. Proceeding in the same way with columns 2 to 5 results in the rectangles in figure 3.51.

8	23	38
15	30	45
1	16	31

a) S_1

12	27	42
2	17	32
10	25	40

b) S_2

3	18	33
7	22	37
14	29	44

c) S_3

4	19	34
11	26	41
9	24	39

d) S_4

13	28	43
5	20	35
6	21	36

e) S_5

Figure 3.51: Construction of rectangles S_1 to S_5

The numbers are again taken diagonally from these rectangles S_1 to S_5 and entered into the rows of the corresponding rectangles R_i. These are shown in figure 3.52.

8	30	31
45	1	23
16	38	15

a) R_1

12	17	40
32	10	27
25	42	2

b) R_2

3	22	44
37	14	18
29	33	7

c) R_3

4	26	39
41	9	19
24	34	11

d) R_4

13	20	36
35	6	28
21	43	5

e) R_5

Figure 3.52: Transferring the numbers from S_i to R_i

All obtained rectangles R_i have the same row and column sums 69. If you put them together into a rectangle $R(3, 15)$, this rectangle must be magic as in figure 3.53. The row sums are $5 \cdot 69 = 345$ and the column sums are 69.

8	30	31	12	17	40	3	22	44	4	26	39	13	20	36
45	1	23	32	10	27	37	14	18	41	9	19	35	6	28
16	38	15	25	42	2	29	33	7	24	34	11	21	43	5
	R_1			R_2			R_3			R_4			R_5	

Figure 3.53: Magic rectangle of size 3 x 15 (Bier - Rogers)

3.4 Chai - Das - Midha

Chai, Das and Midha presented a method to create magic rectangles $R(m, n)$ when the number of rows and columns is odd, and in addition, n is not a multiple of 3.[17] As with all presented methods here, the origin $(1, 1)$ of the coordinate system is the upper left corner. Their deviating designations with p for the number of rows and q for the columns are retained here, since otherwise a comparison with their original article would be very difficult due to the variety of matrices.

For their construction, they first define some matrices where $q = 2q' - 1$ with $q' \geq 1$ applies.

[17] Chai - Das - Midha [4]

$$S = \begin{pmatrix} 1 & 2 & \cdots & q' & q'+1 & \cdots & q \\ q' & q'+1 & \cdots & q & 1 & \cdots & q'-1 \\ q & q-2 & \cdots & 1 & q-1 & \cdots & 2 \end{pmatrix}$$

$$\begin{pmatrix} R_u \\ R_l \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & q-1 & q \\ q & q-1 & \cdots & 2 & 1 \end{pmatrix}$$

$$T_u = \begin{pmatrix} q & q-2 & \cdots & 1 & q-1 & q-3 & \cdots & 2 \\ 1 & 2 & \cdots & q' & q'+1 & q'+2 & \cdots & q \end{pmatrix}$$

$$T_l = \begin{pmatrix} 1 & 3 & \cdots & q & 2 & 4 & \cdots & q-1 \\ q & q-1 & \cdots & q' & q'-1 & q'-2 & \cdots & 1 \end{pmatrix}$$

Table 14: Definition of the matrices S, R_u, R_l, T_u and T_l

This means that all rows of these matrices are permutations of the numbers $1, 2, \ldots, q$, where the column sums of R, S, T_u and T_l are $q+1$, $3\,\frac{q+1}{2}$, $q+1$ and $q+1$.

In addition to these matrices, another matrix G is required, whose definition distinguishes several cases.

$$p = 3 \qquad G_3 = S$$

$$p = 5 \qquad G_5 = \begin{pmatrix} R_u \\ S \\ R_l \end{pmatrix}$$

$$p > 5 \qquad G_p = \begin{cases} \begin{pmatrix} E_{p'1} \otimes T_u \\ S \\ E_{p'1} \otimes T_l \end{pmatrix} & \text{for } p = 4p'+3 \\ \begin{pmatrix} R_u \\ G_{p-2} \\ R_l \end{pmatrix} & \text{for } p = 4p'+5 \end{cases}$$

Table 15: Definition of matrix G_p

Here, $E_{p'1}$ means a matrix with p' rows and a single column. The operator $\otimes$ represents the well-known Kronecker product in mathematics. For some values of p' the result of the Kronecker product of the two matrices $E_{p'1}$ and T_u is given in table 16, so that the systematic in these special cases of the Kronecker product can be recognized.

p'	$E_{p'1} \otimes T_u$	Kronecker product
1	$\begin{pmatrix} 1 \end{pmatrix} \otimes \begin{pmatrix} T_u \end{pmatrix}$	$\begin{pmatrix} T_u \end{pmatrix}$
2	$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} T_u \end{pmatrix}$	$\begin{pmatrix} T_u \\ T_u \end{pmatrix}$
3	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} T_u \end{pmatrix}$	$\begin{pmatrix} T_u \\ T_u \\ T_u \end{pmatrix}$

Table 16: Kronecker product

The value p' thus indicates how often the matrix T_u and thus also T_l must be duplicated in each case. For $p = 4 \cdot 1 + 3 = 7$ and thus $p' = 1$, the following matrices result for G_7, G_9, G_{11} and G_{13}:

$$G_7 = \begin{pmatrix} T_u \\ S \\ T_l \end{pmatrix} \qquad G_9 = \begin{pmatrix} R_u \\ G_7 \\ R_l \end{pmatrix} \qquad G_{11} = \begin{pmatrix} T_u \\ T_u \\ S \\ T_l \\ T_l \end{pmatrix} \qquad G_{13} = \begin{pmatrix} R_u \\ G_9 \\ R_l \end{pmatrix}$$

The matrix G is transformed into a matrix H, where the following expression for $1 \leq i \leq p$ and $1 \leq j \leq q$ applies:

$$h_{ij} = \left(g_{ij} - 1\right) \cdot p + i$$

The matrix H has the following properties:

1. each of the numbers $1, 2, \ldots, pq$ occur exactly once
2. the column sums are $p \cdot \frac{pq+1}{2}$
3. the row sum of a row i is $pq \cdot \frac{q-1}{2} + qi$
4. for $1 \leq i \leq \frac{p-1}{2}$ the difference between rows i and $p + 1 - i$ is always $q \cdot (p + 1 - 2i)$
5. the row sum of the center row $\frac{p+1}{2}$ has the required row sum $q \cdot (\frac{pq+1}{2})$ for a magic rectangle of this size

Finally, the matrix H is transformed into a magic rectangle by swapping components. These swaps can be divided into four cases, where $l = \frac{q-p}{2}$ applies.

1. $\frac{p-5}{4} = \left\lfloor \frac{p-5}{4} \right\rfloor$

   First, a parameter y is set, which is used in this and also in the second case.

$$y = \left\lfloor \frac{l-1}{2} \right\rfloor$$

   Then some swaps are performed:

for $1 \leq j \leq y$	$h_{1,j}$ $h_{1,q+1-j}$	$\longleftrightarrow$ $\longleftrightarrow$	$h_{p,j}$ $h_{p,q+1-j}$
for $l = 2y + 1$	$h_{1,\frac{2q+3-p}{4}}$	$\longleftrightarrow$	$h_{p,\frac{2q+3-p}{4}}$
for $l = 2y + 2$	$h_{1,\frac{2q+3-p}{4}}$ $h_{1,\frac{q+1}{2}}$	$\longleftrightarrow$ $\longleftrightarrow$	$h_{p,\frac{2q+3-p}{4}}$ $h_{p,\frac{q+1}{2}}$

Table 17: Swapping numbers in case 1

2. The parameter y is set as in case 1. Then, in this case, numbers in the rows i and $p+1-i$ are swapped, where i is defined by the interval

$$\frac{p-1}{2} - 2 \cdot \left\lfloor \frac{p-3}{4} \right\rfloor \leq i \leq \frac{p-3}{2}$$

Depending on y, the following pairs of numbers are affected:

for $y > 0$ and $1 \leq j \leq y$	$h_{i,\frac{q+3}{2}-j}$ $h_{i,\frac{q+1}{2}+j}$	$\longleftrightarrow$ $\longleftrightarrow$	$h_{p+1-j,\frac{q+3}{2}-j}$ $h_{p+1-j,\frac{q+1}{2}+j}$
for $l = 2y+1$	$h_{i,\frac{p+3}{2}-i}$	$\longleftrightarrow$	$h_{p+1-i,\frac{p+3}{2}-i}$
for $l = 2y+2$	$h_{i,1}$ $h_{i,\frac{p+3}{2}-i}$	$\longleftrightarrow$ $\longleftrightarrow$	$h_{p+1-i,1}$ $h_{p+1-i,\frac{p+3}{2}-i}$

Table 18: Swapping numbers in case 2

3. The third case is defined by the equality $\frac{q+1}{6} = \left\lfloor \frac{q+1}{6} \right\rfloor$. In this case, numbers in rows $\frac{p-1}{2}$ and $\frac{p+3}{2}$ are swapped. y is calculated in this and also in the fourth case by

$$y = \left\lfloor \frac{l-1}{4} \right\rfloor$$

for $y > 0$ and $1 \leq j \leq y$	$h_{\frac{p-1}{2},j}$ $h_{\frac{p-1}{2},\frac{q+1}{6}+j}$ $h_{\frac{p-1}{2},\frac{q+3}{2}-j}$ $h_{\frac{p-1}{2},(q+1)-j}$	$\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$	$h_{\frac{p+3}{2},j}$ $h_{\frac{p+3}{2},\frac{q+1}{6}+j}$ $h_{\frac{p+3}{2},\frac{q+3}{2}-j}$ $h_{\frac{p+3}{2},(q+1)-j}$
for $l = 4y+1$	$h_{\frac{p-1}{2},\frac{q+1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+1}{3}}$
for $l = 4y+2$	$h_{\frac{p-1}{2},\frac{q+1}{3}}$ $h_{\frac{p-1}{2},2\frac{q+1}{3}}$	$\longleftrightarrow$ $\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+1}{3}}$ $h_{\frac{p+3}{2},2\frac{q+1}{3}}$
for $l = 4y+3$	$h_{\frac{p-1}{2},\frac{q+1}{3}}$ $h_{\frac{p-1}{2},\frac{2q-1}{3}}$ $h_{\frac{p-1}{2},\frac{2q+5}{3}}$	$\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+1}{3}}$ $h_{\frac{p+3}{2},\frac{2q-1}{3}}$ $h_{\frac{p+3}{2},\frac{2q+5}{3}}$
for $l = 4y+4$	$h_{\frac{p-1}{2},\frac{q+1}{3}}$ $h_{\frac{p-1}{2},\frac{2q-1}{3}}$ $h_{\frac{p-1}{2},\frac{2q+5}{3}}$ $h_{\frac{p-1}{2},2\frac{q+1}{3}}$	$\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+1}{3}}$ $h_{\frac{p+3}{2},\frac{2q-1}{3}}$ $h_{\frac{p+3}{2},\frac{2q+5}{3}}$ $h_{\frac{p+3}{2},2\frac{q+1}{3}}$

Table 19: Swapping numbers in case 3

4. The fourth and last case occur, when $\frac{q-1}{6} = \left\lfloor \frac{q-1}{6} \right\rfloor$ holds. Then again, numbers in rows $\frac{p-1}{2}$ and $\frac{p+3}{2}$ are swapped.

for $y > 0$ and $1 \le j \le y$	$h_{\frac{p-1}{2},j}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},j}$
	$h_{\frac{p-1}{2},\frac{q+1}{2}+j}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+1}{2}+j}$
	$h_{\frac{p-1}{2},\frac{5q+7}{6}-j}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{5q+7}{6}-j}$
	$h_{\frac{p-1}{2},(q+1)-j}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},(q+1)-j}$
for $l = 4y+1$	$h_{\frac{p-1}{2},\frac{2q+1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{2q+1}{3}}$
for $l = 4y+2$	$h_{\frac{p-1}{2},\frac{2q+1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{2q+1}{3}}$
	$h_{\frac{p-1}{2},\frac{q+2}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+2}{3}}$
for $l = 4y+3$	$h_{\frac{p-1}{2},\frac{2q+1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{2q+1}{3}}$
	$h_{\frac{p-1}{2},\frac{q-1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q-1}{3}}$
	$h_{\frac{p-1}{2},\frac{q+5}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+5}{3}}$
for $l = 4y+4$	$h_{\frac{p-1}{2},\frac{2q+1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{2q+1}{3}}$
	$h_{\frac{p-1}{2},\frac{q-1}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q-1}{3}}$
	$h_{\frac{p-1}{2},\frac{q+5}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+5}{3}}$
	$h_{\frac{p-1}{2},\frac{q+2}{3}}$	$\longleftrightarrow$	$h_{\frac{p+3}{2},\frac{q+2}{3}}$

Table 20: Swapping numbers in case 4

Altogether, these are many apparently complex swaps, but they can be well implemented by an algorithm.

Example 9x13

As a first example, a magic rectangle $R(9,13)$ shall be generated, resulting in $l = \frac{13-9}{2} = 2$. With $p = 4 \cdot 1 + 5 = 9$ the lower case in the definition of G_p is valid for G_9. Thus, first G_7 must be calculated to fill the two upper and lower rows of the target rectangle with T_u and T_l.

13	11	9	7	5	3	1	12	10	8	6	4	2	T_u
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	2	3	4	5	6	7	8	9	10	11	12	13	S
7	8	9	10	11	12	13	1	2	3	4	5	6	
13	11	9	7	5	3	1	12	10	8	6	4	2	
1	3	5	7	9	11	13	2	4	6	8	10	12	T_l
13	12	11	10	9	8	7	6	5	4	3	2	1	

Figure 3.54: Matrix G_7

G_7 can now be used to determine the matrix G_9 from figure 3.55.

1	2	3	4	5	6	7	8	9	10	11	12	13	R_u
13	11	9	7	5	3	1	12	10	8	6	4	2	G_7
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	2	3	4	5	6	7	8	9	10	11	12	13	
7	8	9	10	11	12	13	1	2	3	4	5	6	
13	11	9	7	5	3	1	12	10	8	6	4	2	
1	3	5	7	9	11	13	2	4	6	8	10	12	
13	12	11	10	9	8	7	6	5	4	3	2	1	
13	12	11	10	9	8	7	6	5	4	3	2	1	R_l

Figure 3.55: Matrix G_9

Then G_9 is transformed into the matrix H_9.

$$h_{ij} = (g_{ij} - 1) \cdot p + i$$

H_9 already has the desired column sums 531 and in addition also the center row already has the row sum 767.

1	10	19	28	37	46	55	64	73	82	91	100	109
110	92	74	56	38	20	2	101	83	65	47	29	11
3	12	21	30	39	48	57	66	75	84	93	102	111
4	13	22	31	40	49	58	67	76	85	94	103	112
59	68	77	86	95	104	113	5	14	23	32	41	50
114	96	78	60	42	24	6	105	87	69	51	33	15
7	25	43	61	79	97	115	16	34	52	70	88	106
116	107	98	89	80	71	62	53	44	35	26	17	8
117	108	99	90	81	72	63	54	45	36	27	18	9

Figure 3.56: Matrix H_9

Now the four cases for swaps are checked one after the other. The conditions for case 1 apply in this example, and y is calculated as $y = \lfloor \frac{l-1}{2} \rfloor = 0$. Of the three possible conditions for this case, $l = 2y + 2$ applies and two swaps are performed in rows 1 and p.

$$h_{1,\frac{2q+3-p}{4}} \longleftrightarrow h_{p,\frac{2q+3-p}{4}} \qquad h_{1,5} \longleftrightarrow h_{9,5}$$

$$h_{1,\frac{q+1}{2}} \longleftrightarrow h_{p,\frac{q+1}{2}} \qquad h_{1,7} \longleftrightarrow h_{9,7}$$

This exchanges the numbers in rows 1 and 9 and columns 5 and 7 vertically symmetrically. [18]

The condition for the second case of exchanges is also satisfied. Here, numbers from 1 and $p+1-i$ are exchanged, whereby the limits of the range for i can be calculated according to the given formula.

$$\begin{array}{rcccl} \frac{p-1}{2} - 2 \cdot \left\lfloor \frac{p-3}{4} \right\rfloor & \leq & i & \leq & \frac{p-3}{2} \\ 4 - 2 \cdot 1 & \leq & i & \leq & 3 \\ 2 & \leq & i & \leq & 3 \end{array}$$

Rows 2 and 3 are affected. With $l = 2y + 2$, this results in a total of four exchanges.[19]

$$h_{i,1} \longleftrightarrow h_{p+1-i,1} \qquad h_{2,1} \longleftrightarrow h_{8,1}, \quad h_{2,4} \longleftrightarrow h_{8,4}$$

$$h_{i,\frac{p+3}{2}-i} \longleftrightarrow h_{p+1-i,\frac{p+3}{2}-i} \qquad h_{3,1} \longleftrightarrow h_{7,1}, \quad h_{3,3} \longleftrightarrow h_{7,3}$$

Furthermore, the condition for the fourth case of exchanges is also satisfied. With $l = 4y+2$, two further exchanges must be carried out.[20]

$$h_{\frac{p-1}{2},\frac{2q+1}{3}} \longleftrightarrow h_{\frac{p+3}{2},\frac{2q+1}{3}} \qquad h_{4,9} \longleftrightarrow h_{6,9}$$

$$h_{\frac{p-1}{2},\frac{q+2}{3}} \longleftrightarrow h_{\frac{p+3}{2},\frac{q+2}{3}} \qquad h_{4,5} \longleftrightarrow h_{6,5}$$

With these exchanges, the magic rectangle $R(9, 13)$ is created, which is shown in figure 3.57. The row sums are 767 and the column sums are 531.

1	10	19	28	81	46	63	64	73	82	91	100	109
116	92	74	89	38	20	2	101	83	65	47	29	11
7	12	43	30	39	48	57	66	75	84	93	102	111
4	13	22	31	42	49	58	67	87	85	94	103	112
59	68	77	86	95	104	113	5	14	23	32	41	50
114	96	78	60	40	24	6	105	76	69	51	33	15
3	25	21	61	79	97	115	16	34	52	70	88	106
110	107	98	56	80	71	62	53	44	35	26	17	8
117	108	99	90	37	72	55	54	45	36	27	18	9

Figure 3.57: Magic rectangle of size 9 x 13 (Chai-Das-Midha)

[18] in figure 3.57 these cells are colored green

[19] in figure 3.57 these cells are colored blue

[20] in figure 3.57 these cells are colored red

Example 11x19

As a second example, the magic rectangle $R(11, 19)$ is to be created so that the exchanges become even more understandable. With $p = 11$ and $q = 19$ follows $l = \frac{q-p}{2} = \frac{19-11}{2} = 4$ and G_{11} must be determined using the first case of the definition of G_p.

19	17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	6	4	2
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
19	17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	6	4	2
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9
19	17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	6	4	2
1	3	5	7	9	11	13	15	17	19	2	4	6	8	10	12	14	16	18
19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
1	3	5	7	9	11	13	15	17	19	2	4	6	8	10	12	14	16	18
19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Figure 3.58: Matrix G_{11}

Then G_{11} is transformed into the matrix H_{11}. As always, this matrix has already the desired column sums 1155 and additionally also the center row has the row sum 1995.

199	177	155	133	111	89	67	45	23	1	188	166	144	122	100	78	56	34	12
2	13	24	35	46	57	68	79	90	101	112	123	134	145	156	167	178	189	200
201	179	157	135	113	91	69	47	25	3	190	168	146	124	102	80	58	36	14
4	15	26	37	48	59	70	81	92	103	114	125	136	147	158	169	180	191	202
5	16	27	38	49	60	71	82	93	104	115	126	137	148	159	170	181	192	203
105	116	127	138	149	160	171	182	193	204	6	17	28	39	50	61	72	83	94
205	183	161	139	117	95	73	51	29	7	194	172	150	128	106	84	62	40	18
8	30	52	74	96	118	140	162	184	206	19	41	63	85	107	129	151	173	195
207	196	185	174	163	152	141	130	119	108	97	86	75	64	53	42	31	20	9
10	32	54	76	98	120	142	164	186	208	21	43	65	87	109	131	153	175	197
209	198	187	176	165	154	143	132	121	110	99	88	77	66	55	44	33	22	11

Figure 3.59: Matrix H_{11}

It is now checked, which conditions apply to the four cases of the swapping to be carried out. In this example, case 1 is skipped, and you start with case 2, where $y = \lfloor \frac{l-1}{2} \rfloor = 1$ applies. For the involved rows i the range

$$\begin{aligned}\frac{p-1}{2} - 2\cdot\left\lfloor\frac{p-3}{4}\right\rfloor \;&\le\; i \;\le\; \frac{p-3}{2}\\ 5-2\cdot 2 \;&\le\; i \;\le\; 4\\ 1 \;&\le\; i \;\le\; 4\end{aligned}$$

results. Thus, the first condition is true, and eight swaps are carried for $1 \le i \le 4$.

$$\begin{array}{lcl@{\qquad\qquad}lcl} h_{1,10} & \longleftrightarrow & h_{11,10} & h_{3,10} & \longleftrightarrow & h_{9,10}\\ h_{1,11} & \longleftrightarrow & h_{11,11} & h_{3,11} & \longleftrightarrow & h_{9,11}\\ h_{2,10} & \longleftrightarrow & h_{10,10} & h_{4,10} & \longleftrightarrow & h_{8,10}\\ h_{2,11} & \longleftrightarrow & h_{10,11} & h_{4,11} & \longleftrightarrow & h_{8,11}\end{array}$$

In addition, the third condition $l = 2y + 2$ also applies here, so that four more swaps must be carried out.[21]

$$\begin{array}{lcl@{\qquad\qquad}lcl} h_{1,1} & \longleftrightarrow & h_{11,1} & h_{3,1} & \longleftrightarrow & h_{9,1}\\ h_{1,6} & \longleftrightarrow & h_{11,6} & h_{3,4} & \longleftrightarrow & h_{9,4}\\ h_{2,1} & \longleftrightarrow & h_{10,1} & h_{4,1} & \longleftrightarrow & h_{8,1}\\ h_{2,5} & \longleftrightarrow & h_{10,5} & h_{4,3} & \longleftrightarrow & h_{8,3}\end{array}$$

In addition to case 2, case 4 also satisfies the specified conditions, so that four additional swaps have to be carried out with $l = 4y + 4$.

$$\begin{array}{lcl@{\qquad\qquad}lcl} h_{5,13} & \longleftrightarrow & h_{7,13} & h_{5,8} & \longleftrightarrow & h_{7,8}\\ h_{5,6} & \longleftrightarrow & h_{7,6} & h_{5,7} & \longleftrightarrow & h_{7,7}\end{array}$$

Overall, these swaps yield the magic rectangle $R(11, 19)$ from figure 3.60. The row sums are 1995 and the column sums 1155.

209	177	155	133	111	154	67	45	23	110	99	166	144	122	100	78	56	34	12
10	13	24	35	98	57	68	79	90	208	21	123	134	145	156	167	178	189	200
207	179	157	174	113	91	69	47	25	108	97	168	146	124	102	80	58	36	14
8	15	52	37	48	59	70	81	92	206	19	125	136	147	158	169	180	191	202
5	16	27	38	49	95	73	51	93	104	115	126	150	148	159	170	181	192	203
105	116	127	138	149	160	171	182	193	204	6	17	28	39	50	61	72	83	94
205	183	161	139	117	60	71	82	29	7	194	172	137	128	106	84	62	40	18
4	30	26	74	96	118	140	162	184	103	114	41	63	85	107	129	151	173	195
201	196	185	135	163	152	141	130	119	3	190	86	75	64	53	42	31	20	9
2	32	54	76	46	120	142	164	186	101	112	43	65	87	109	131	153	175	197
199	198	187	176	165	89	143	132	121	1	188	88	77	66	55	44	33	22	11

Figure 3.60: Magic rectangle of size 11 x 19 (Chai-Das-Midha)

[21] in figure 3.57, all cells for this case 2 are colored green

4 Bordered magic rectangles

A *bordered magic rectangle* is a magic rectangle that remains magic, even if you remove its border. If you remove the left and right columns and the top and bottom rows in the 10 x 12 magic rectangle shown in figure 4.1, you get a new magic rectangle of size 8 x 10.

1	119	118	4	5	115	114	8	9	111	110	12
13	29	21	99	98	24	25	95	94	28	92	108
107	91	37	83	82	40	41	79	78	44	30	14
106	90	45	53	49	71	70	52	68	76	31	15
16	32	75	67	57	63	62	60	54	46	89	105
17	33	74	66	64	58	59	61	55	47	88	104
103	87	48	56	72	50	51	69	65	73	34	18
102	86	84	38	39	81	80	42	43	77	35	19
20	36	100	22	23	97	96	26	27	93	85	101
120	2	3	117	116	6	7	113	112	10	11	109

29	21	99	98	24	25	95	94	28	92
91	37	83	82	40	41	79	78	44	30
90	45	53	49	71	70	52	68	76	31
32	75	67	57	63	62	60	54	46	89
33	74	66	64	58	59	61	55	47	88
87	48	56	72	50	51	69	65	73	34
86	84	38	39	81	80	42	43	77	35
36	100	22	23	97	96	26	27	93	85

Figure 4.1: Bordered magic rectangle of size 10 x 12

Although this rectangle is no longer normalized, i.e. it no longer contains the numbers $1, 2, \ldots, m\,n$. But it still has the property that all row sums are identical as well as all column sums.

As a second condition for a bordered magic rectangle it is required that the border numbers enclose the numbers of the inner magic rectangle. For a rectangle $R(m, n)$ this means that a total of $2 \cdot (m + n - 2)$ numbers lie in the border and enclose the other numbers. Half of these numbers, i.e. $m + n - 2$, lying in the border form the lower border numbers, the other half the upper border numbers. Thus, the numbers of a border are divided as shown in table 21:

lower border numbers:	$1 \;\ldots\; m + n - 2$
inner numbers:	$m + n - 2 + 1 \;\ldots\; m\,n - (m + n - 2)$
upper border numbers:	$m\,n - (m + n - 2) + 1 \;\ldots\; m\,n$

Table 21: Division of numbers in a border

Multi-bordered magic rectangles

If you take a closer look at the rectangle in figure 4.1, you will notice that by continuing to remove the outer border, you will obtain more and more magic rectangles. Such a rectangle, like the one in figure 4.2, is called a *multi-bordered magic rectangle*.

1	119	118	4	5	115	114	8	9	111	110	12
13	29	21	99	98	24	25	95	94	28	92	108
107	91	37	83	82	40	41	79	78	44	30	14
106	90	45	53	49	71	70	52	68	76	31	15
16	32	75	67	57	63	62	60	54	46	89	105
17	33	74	66	64	58	59	61	55	47	88	104
103	87	48	56	72	50	51	69	65	73	34	18
102	86	84	38	39	81	80	42	43	77	35	19
20	36	100	22	23	97	96	26	27	93	85	101
120	2	3	117	116	6	7	113	112	10	11	109

Figure 4.2: Multi-bordered magic rectangle of size 10 x 12

But it should also be mentioned that sometimes one deviates a little from the strict conditions for bordered magic rectangles and neglects the condition for the division of the numbers. The only condition that remains then is that removing the outer rows and columns creates a magic rectangle again. Such magic rectangles are called *concentric*.

This division into bordered and concentric rectangles did not always exist in the past. But it has become established in recent decades.

In this chapter, I present some techniques to create bordered magic rectangles. I am not aware of any article in the literature that describes the construction of such magic rectangles.

4.1 m and n are of different types

This section describes the construction of a magic rectangle $R(m, n)$, when the number m of rows is single-even and the number n of columns n is double-even or vice versa and $m, n \geq 4$ holds. The construction is done from the outside to the inside, adding another border with each step. To define different sizes clearly, the two initial sizes are marked with m and n, while the individual borders always receive the border number as associated index. Thus, for the first border applies

$$m_1 = m \qquad \text{and} \qquad n_1 = n$$

In the first example, a magic rectangle $R(6, 8)$ will be created so that the number of columns $n_1 = 8$ is double-even. For this purpose, a magic rectangle $R_1(2, n_1)$ is created first. To allow subsequent borders, all numbers of this initial rectangle greater than $\frac{n_1}{2} = \frac{8}{2} = 4$ are increased by

$$m_1 n_1 - 2n_1 = 6 \cdot 8 - 2 \cdot 8 = 32$$

and the two rows of R_1 are inserted into the top and bottom rows of the target rectangle as shown in figure 4.3.

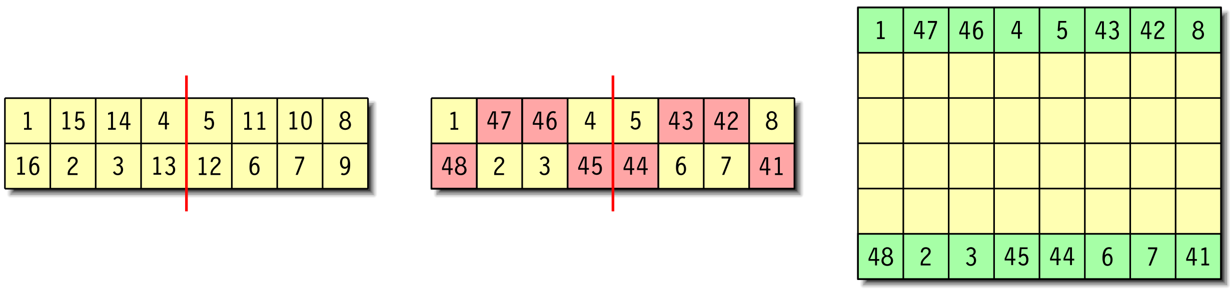


Figure 4.3: Entering numbers in the rows of the first border

This splits the first numbers of the rectangle as in table 22.

lower border numbers	upper border numbers
$1 \ldots n_1$	$m_1 n_1 - n_1 + 1 \ldots m_1 n_1$
$1 \ldots 8$	$41 \ldots 48$

Table 22: First numbers for the outer border

Another rectangle $R_2(m_1 - 2, 2)$ is required for the two outer columns of this border, which can easily be generated, since $m_1 - 2 = 4$ is also double-even. The smaller $m_1 - 2$ numbers of the initial rectangle R_2 are each increased by $n_1 = 8$, so that they directly follow the already existing lower border numbers. Correspondingly, the larger numbers of R_2 are increased by

$$m_1 n_1 - n_1 - 2 \cdot (m_1 - 2) = 6 \cdot 8 - 8 - 2 \cdot (6 - 2) = 32$$

so that they connect directly to the lower end of the upper border numbers. If you insert the columns of this rectangle into the two outer columns of the target rectangle, you get the completely filled first border of size 6 x 8.

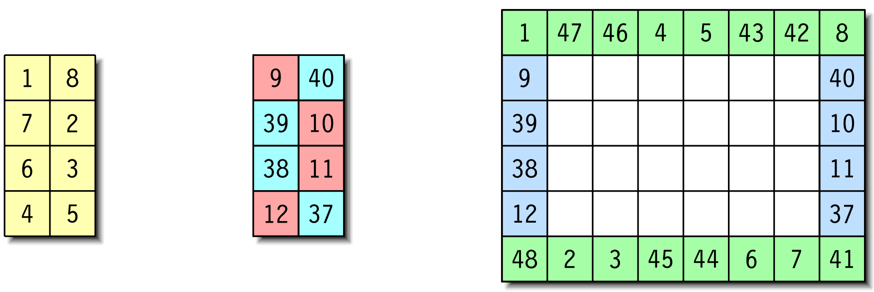


Figure 4.4: First border of size 6 x 8

Thus, all required border numbers have been placed in the first border as shown in figure 4.4. The distribution of these numbers is shown in table 23. The middle 24 numbers not yet used are then available for subsequent borders.

	lower border numbers	upper border numbers
rows	1 . . . 8	41 . . . 48
columns	9 . . . 12	37 . . . 40
total	1 . . . 12	37 . . . 48

Table 23: Numbers of the first border

The next border of size 4 x 6 is basically created in the same way. However, since the number of rows $m_2 = 4$ is double-even here, a magic rectangle $R_3(2, 4)$ is first generated that does not cover the entire upper row. As with the previous border, the numbers in the upper half of this rectangle are incremented, but according to table 26, a different formula is used for the calculation.

$$m_2 n_2 - 2 \cdot (n_2 - 2) = 4 \cdot 6 - 2 \cdot (6 - 2) = 16$$

Then the two rows of this rectangle can be inserted into the center of the upper and lower row of the current border.

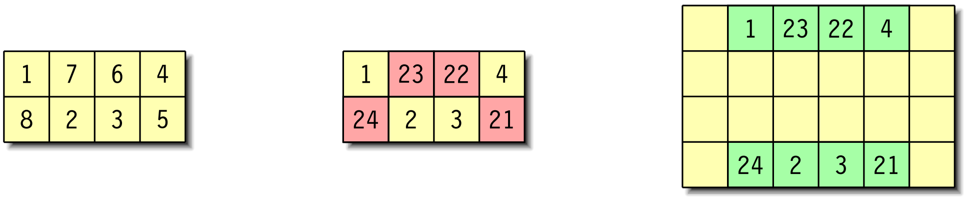


Figure 4.5: Entering the numbers into the rows of the second border

The border numbers now include the ranges $1, \ldots, 4$ and $21, \ldots, 24$.

For the two outer columns of this border another rectangle $R_4(4, 2)$ is created, whose smaller numbers are increased by $n_2 - 2 = 4$. With such an arrangement, a different formula must also be used for the calculation to increase the larger numbers of R_4, so that they connect to the already existing number ranges of the border numbers.

With this step, the border numbers from table 24 result for this border and the border of size 4 x 6 is uniquely determined.

	lower border numbers	upper border numbers
rows	1 . . . 4	21 . . . 24
columns	5 . . . 8	17 . . . 20
total	1 . . . 8	17 . . . 24

Table 24: Preliminary numbers for the second border

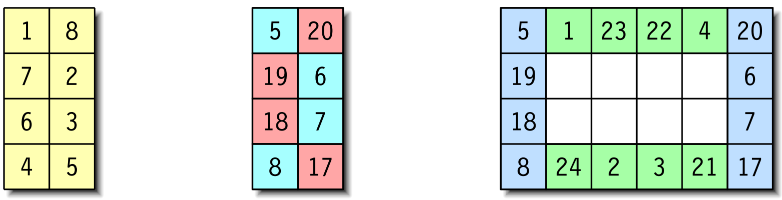


Figure 4.6: Basic numbers for the second border

To be able to insert this border into the target rectangle, all numbers in this border must be increased by the constant c

$$c = m_1 + n_1 - 2 = 6 + 8 - 2 = 12$$

so that they do not overlap with numbers of the first border. This completes the division of numbers for this border, which is shown in table 25.

	lower border numbers	upper border numbers
preliminary	$1 \dots 8$	$17 \dots 24$
final	$13 \dots 20$	$29 \dots 36$

Table 25: Numbers for the second border

If you insert this border into the target rectangle, the two outer borders are already filled and shown in figure 4.7.

17	13	35	34	16	32
31					18
30					19
20	36	14	15	33	29

1	47	46	4	5	43	42	8
9	17	13	35	34	16	32	40
39	31					18	10
38	30					19	11
12	20	36	14	15	33	29	37
48	2	3	45	44	6	7	41

Figure 4.7: Second border of size 4 x 6

For rectangles of the types discussed in this section, another border of size 2 x 4 can be inserted, so that a magic rectangle $R_5(2, 4)$ of this size is created now. In addition, the increment number c has to be adjusted for the last border

$$c = c + m_2 + n_2 - 2 = 12 + 4 + 6 - 2 = 20$$

so that all numbers of R_5 are now increased by $c = 20$.

1	7	6	4
8	2	3	5

21	27	26	24
28	22	23	25

Figure 4.8: Center of the magic rectangle

If you finally insert this border into the target rectangle, the multi-bordered magic rectangle of size 6 x 8 from figure 4.9 with the row sums 196 and the column sums 147 is created.

1	47	46	4	5	43	42	8
9	17	13	35	34	16	32	40
39	31	21	27	26	24	18	10
38	30	28	22	23	25	19	11
12	20	36	14	15	33	29	37
48	2	3	45	44	6	7	41

Figure 4.9: Multi-bordered magic rectangle of size 6 x 8

With this example it has become clear that the constant c to increase the numbers of the initial rectangles $R(2, 4k)$ and $R(4k, 2)$ depends on the number of columns of the current border. Therefore, the corresponding formulas for the calculation are summarized in table 26.

columns	horizontally		vertical	
	smaller numbers	larger numbers	smaller numbers	larger numbers
$4k$	0	$mn - 2n$	n	$mn - n - 2 \cdot (m-2)$
$4k+2$	0	$mn - 2 \cdot (n-2)$	$n-2$	$mn - (n-2) - 2m$

Table 26: Increment value for the initial rectangles

Of course, bordered magic rectangles can also be created with this method, if $m > n$ holds for the size.

Example 2

A second example shall illustrate the construction of a multi-bordered magic rectangle $R(8, 14)$, where additionally also possible variations are given. First of all, an initial rectangle $R_1(2, 12)$ is required for the rows. In the first example, groups of four numbers were always placed in a special order, until the desired number of columns was reached.

However, this is not necessary, since any rectangles of size 2 x 4k can also be used. All you have to do is to enter the first k consecutive numbers in one of the two rows, followed by $2k$ numbers in the other row and finally k numbers again in the first selected row. Thus, for 12 columns four possible arrangements of numbers for magic rectangles $R(2, 12)$ exist, which are shown in figure 4.10, where the separations between the groups are especially marked.

1	23	22	4	5	19	18	8	9	15	14	12
24	2	3	21	20	6	7	17	16	10	11	13

a) three groups $R(2, 4)$

1	23	22	4	5	6	18	17	16	15	11	12
24	2	3	21	20	19	7	8	9	10	14	13

b) $R(2, 4)$ and $R(2, 8)$

1	2	22	21	20	19	7	8	9	15	14	12
24	23	3	4	5	6	18	17	16	10	11	13

c) $R(2, 8)$ and $R(2, 4)$

1	2	3	21	20	19	18	17	16	10	11	12
24	23	22	4	5	6	7	8	9	15	14	13

d) $R(2, 12)$

Figure 4.10: Some possible variations for the magic rectangle $R(2, 12)$

Of course, there are other variations, since you can decide, for example, for each individual group, whether you start in the top or bottom row. In addition, the columns of the groups can be permuted independently of each other. The only thing that must be ensured here, as in figure 4.11, is that the upper and lower numbers assigned to each other are not changed.

21	10	11	19	12	3	16	2	17	20	1	18
4	15	14	6	13	22	9	23	8	5	24	7

Figure 4.11: Rectangle $R(2, 12)$ with permuted columns

The number of columns for the outer border of the magic rectangle $R(8, 14)$ is single-even, so that first a rectangle of size 2×12 must be determined. For the division of the columns the variant from figure 4.10c has been chosen. After increasing the numbers in the two initial rectangles for the two rows and columns, the border from figure 4.12 is created.

13	1	2	110	109	108	107	7	8	9	103	102	12	100
99													14
98													15
16													97
17													96
95													18
94													19
20	112	111	3	4	5	6	106	105	104	10	11	101	93

Figure 4.12: First border of size 8×14

For the next border, two initial rectangles $R(2, 12)$ are needed for the rows and $R(4, 2)$ for the columns.

1	71	70	4	68	67	7	8	9	10	62	61
13											60
59											14
58											15
16											57
72	2	3	69	5	6	66	65	64	63	11	12

a) initial rectangles $R(2, 12)$ and $R(4, 2)$

21	91	90	24	88	87	27	28	29	30	82	81
33											80
79											34
78											35
36											77
92	22	23	89	25	26	86	85	84	83	31	32

b) increased numbers in $R(2, 12)$ and $R(4, 2)$

13	1	2	110	109	108	107	7	8	9	103	102	12	100
99	21	91	90	24	88	87	27	28	29	30	82	81	14
98	33											80	15
16	79											34	97
17	78											35	96
95	36											77	18
94	92	22	23	89	25	26	86	85	84	83	31	32	19
20	112	111	3	4	5	6	106	105	104	10	11	101	93

Figure 4.13: Second border of size 6 x 12

For the third border, the number of columns is single-even again, so two initial rectangles $R(2, 8)$ and $R(4, 2)$ are required.

9	1	39	38	4	5	35	34	8	32
31									10
30									11
12	40	2	3	37	36	6	7	33	29

a) initial rectangles $R(2, 8)$ and $R(4, 2)$

45	37	75	74	40	41	71	70	44	68
67									46
66									47
48	76	38	39	73	72	42	43	69	65

b) increased numbers in $R(2, 8)$ and $R(4, 2)$

13	1	2	110	109	108	107	7	8	9	103	102	12	100
99	21	91	90	24	88	87	27	28	29	30	82	81	14
98	33	45	37	75	74	40	41	71	70	44	68	80	15
16	79	67									46	34	97
17	78	66									47	35	96
95	36	48	76	38	39	73	72	42	43	69	65	77	18
94	92	22	23	89	25	26	86	85	84	83	31	32	19
20	112	111	3	4	5	6	106	105	104	10	11	101	93

Figure 4.14: Third border of size 4 x 10

Now only the center is empty, which can be filled with an initial rectangle $R(2, 8)$, whose numbers are all increased by 48. This creates the multi-bordered magic rectangle from figure 4.15 with row sums 791 and column sums 452.

1	2	14	13	12	11	7	8
16	15	3	4	5	6	10	9

a) initial rectangle $R(2, 8)$

49	50	62	61	60	59	55	56
64	63	51	52	53	54	58	57

b) increased numbers in $R(2, 8)$

13	1	2	110	109	108	107	7	8	9	103	102	12	100
99	21	91	90	24	88	87	27	28	29	30	82	81	14
98	33	45	37	75	74	40	41	71	70	44	68	80	15
16	79	67	49	50	62	61	60	59	55	56	46	34	97
17	78	66	64	63	51	52	53	54	58	57	47	35	96
95	36	48	76	38	39	73	72	42	43	69	65	77	18
94	92	22	23	89	25	26	86	85	84	83	31	32	19
20	112	111	3	4	5	6	106	105	104	10	11	101	93

Figure 4.15: Multi-bordered magic rectangle of size 8×14

4.2 m and n are of the same type

In this section, multi-bordered magic rectangles are constructed, where the number m of rows and the number n of columns are both double-even or both single-even and $m, n \geq 6$ holds. As a first example, such a rectangle with $m = 6$ and $n = 10$ is generated.

If the width and height of the rectangle are both greater than 4, you start with a fixed given square border of size $m_1 = m = 6$. Such a border with suitable border numbers is shown in figure 4.16a.

1	34	33	32	9	2
10					27
29					8
30					7
6					31
35	3	4	5	28	36

a) square border

1	58	57	56	9	2
10					51
53					8
54					7
6					55
59	3	4	5	52	60

b) partially increased numbers

Figure 4.16: Square border of size 6×6

While the height of this rectangle already corresponds to the desired number of rows, the width of the border must be expanded from the current size $n_1 = 6$ by four additional columns to $n_1 = 10$. To do this, the upper

$$2\,m_1 - 2 = 12 - 2 = 10$$

numbers are first increased by the value

$$m\,n - m_1\,m_1 = 6 \cdot 10 - 6 \cdot 6 = 24$$

With the result in figure 4.16b, the condition for the border numbers is fulfilled.

lower border numbers	upper border numbers
$1 \ldots n_1$	$m_1\,n_1 - n_1 + 1 \ldots m_1\,n_1$
$1 \ldots 10$	$51 \ldots 60$

Table 27: First numbers for the border

Now the four additional columns of the border must be filled. To do this, the right column of the square border is first moved to the right margin of the rectangular border. The four free columns are then filled as usual with the numbers of a rectangle $R(2, 4)$, which are adjacent to the existing ranges of border numbers.

1	58	57	56	9					2
10									51
53									8
54									7
6									55
59	3	4	5	52					60

1	58	57	56	9	11	49	48	14	2
10									51
53									8
54									7
6									55
59	3	4	5	52	50	12	13	47	60

Figure 4.17: Extension of the square border to the size 6 x 10

Thus, the border numbers fulfill the condition for bordered magic rectangles, as can be seen in table 28.

	lower border numbers	upper border numbers
square	$1 \ldots 10$	$51 \ldots 60$
extension	$11 \ldots 14$	$47 \ldots 50$
total	$1 \ldots 14$	$47 \ldots 60$

Table 28: Numbers for the first border

The final step for a rectangle of this size requires a rectangle $R_4(4, 8)$, which is formed from two rectangles of size 2 x 8. The two rows of the first rectangle $R_1(2, 8)$ are transferred to the upper and lower row of the rectangle R_4, where all numbers greater than $n = 10$ are increased by $2\,n = 16$.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

1	31	30	4	5	27	26	8
32	2	3	29	28	6	7	25

Figure 4.18: Outer rows of rectangle $R_4(4, 8)$

Then all numbers of rectangle $R_2(2, 10)$ must be increased by $n = 8$ and are entered into the empty area of R_4.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

9	23	22	12	13	19	18	16
24	10	11	21	20	14	15	17

Figure 4.19: Inner rows of rectangle $R_4(4, 8)$

Finally, all numbers of R_4 are increased by

$$c = m + n - 2 = 6 + 10 - 2 = 14$$

so that they do not overlap with the numbers already entered in the first border.

1	31	30	4	5	27	26	8
9	23	22	12	13	19	18	16
24	10	11	21	20	14	15	17
32	2	3	29	28	6	7	25

15	45	44	18	19	41	40	22
23	37	36	26	27	33	32	30
38	24	25	35	34	28	29	31
46	16	17	43	42	20	21	39

Figure 4.20: Rectangle $R_4(4, 8)$

If you now insert rectangle R_4 into the target rectangle, you get the bordered magic rectangle from figure 4.21 with row sums 305 and column sums 183.

1	58	57	56	9	11	49	48	14	2
10	15	45	44	18	19	41	40	22	51
53	23	37	36	26	27	33	32	30	8
54	38	24	25	35	34	28	29	31	7
6	46	16	17	43	42	20	21	39	55
59	3	4	5	52	50	12	13	47	60

Figure 4.21: Bordered magic rectangle of size 6 x 10

The magic rectangle created is only single-bordered at this size, because only a single border was generated around R_4. In contrast to the procedure from chapter 4.1, the rectangle R_4 cannot be bordered anymore, because another inner rectangle of size 2 x 4k+2 cannot be magic after increasing the numbers.

The method described here is characterized by the fact that you always start with a suitable square border of size 6 x 6 and then extend this border to the right and up. If either $m = 4$ or $n = 4$, the first step with the 6 x 6 - border is omitted and the rectangle $R(4, 4k + 2)$ respectively $R(4k + 2, 4)$ is created directly.

The method obviously also creates bordered magic rectangles, when $m > n$ holds. Only the construction of the inner rectangle $R(4k+2, 4)$ must be adjusted. Another way is to create the rectangle $R(4, 4k+2)$ and then choose the transposed rectangle. For $m = n$, bordered magic squares can generated using this method.

Figure 4.22 on page 77 shows some borders of size 6 x 6 size that can be used as initial borders.

Example 2

For a multi-bordered rectangle with $m = 10$ rows and $n = 14$ columns, start again with a square border of size 6 x 6. Here, the numbers that are greater than 10 must be increased by

$$m\,n - m_1\,m_1 = 10 \cdot 14 - 6 \cdot 6 = 104$$

in the first step.

1	8	31	32	35	4
7					30
9					28
27					10
34					3
33	29	6	5	2	36

a) square border

1	8	135	136	139	4
7					134
9					132
131					10
138					3
137	133	6	5	2	140

b) partially increased numbers

Figure 4.23: Square border of size 6 x 6

With this increment, the first border numbers are already suitably arranged as in figure 4.23. Since several borders are inserted in this example, the height and width receive the border number as associated index for a better distinction. For the first border, $m_1 = m$ and $n_1 = n$ therefore applies.

lower border numbers	upper border numbers
1 . . . 10	131 . . . 140

Table 29: First numbers for the outer border

In this example, the square border must be expanded horizontally by eight columns and vertically by four rows. To do this, the right column of the square border is moved to the right margin of the target rectangle, as in the first example. To create an empty area for the additional rows, the top row of the square border must also be moved up here.

1	9	32	33	34	2
6					31
10					27
29					8
30					7
35	28	5	4	3	36

1	8	31	32	35	4
7					30
9					28
27					10
34					3
33	29	6	5	2	36

1	5	30	34	35	6
8					29
10					27
28					9
33					4
31	32	7	3	2	36

1	3	31	33	35	8
9					28
10					27
30					7
32					5
29	34	6	4	2	36

3	8	30	32	36	2
4					33
10					27
28					9
31					6
35	29	7	5	1	34

3	5	28	35	36	4
6					31
10					27
29					8
30					7
33	32	9	2	1	34

3	4	27	35	36	6
7					30
9					28
29					8
32					5
31	33	10	2	1	34

3	5	28	31	36	8
4					33
10					27
30					7
35					2
29	32	9	6	1	34

5	4	30	34	36	2
6					31
9					28
27					10
29					8
35	33	7	3	1	32

5	2	30	34	36	4
6					31
10					27
28					9
29					8
33	35	7	3	1	32

5	2	28	34	36	6
7					30
8					29
27					10
33					4
31	35	9	3	1	32

5	1	30	33	34	8
6					31
9					28
27					10
35					2
29	36	7	4	3	32

7	3	31	32	36	2
4					33
9					28
27					10
29					8
35	34	6	5	1	30

7	2	28	34	36	4
5					32
8					29
27					10
31					6
33	35	9	3	1	30

7	1	28	34	35	6
5					32
8					29
27					10
33					4
31	36	9	3	2	30

7	5	27	28	36	8
2					35
6					31
33					4
34					3
29	32	10	9	1	30

2	9	30	31	36	3
4					33
10					27
29					8
32					5
34	28	7	6	1	35

2	10	28	32	36	3
6					31
7					30
29					8
33					4
34	27	9	5	1	35

4	3	28	35	36	5
6					31
10					27
29					8
30					7
32	34	9	2	1	33

4	7	29	31	35	5
3					34
9					28
27					10
36					1
32	30	8	6	2	33

Figure 4.22: Some square borders of size 6×6

1	8	135	136	139									4
7													134
9													132
131													10
138													3
137	133	6	5	2									140

Figure 4.24: Extension of the square border to the size 10 x 14

For the horizontal extension, a magic rectangle $R_1(2, 8)$ is created, whose numbers directly follow the previous border numbers and are entered in the upper and lower rows. For the following numbers, another magic rectangle $R_2(4, 2)$ is created, whose numbers are placed into the still empty cells of the two outer columns as in figure 4.25.

1	8	135	136	139	11	12	128	127	126	125	17	18	4
19													122
121													20
120													21
22													119
7													134
9													132
131													10
138													3
137	133	6	5	2	130	129	13	14	15	16	124	123	140

Figure 4.25: Completely filled border of size 10 x 14

This creates the first border from figure 4.25 and the border numbers have been chosen in the individual steps to create a multi-bordered magic rectangle.

	lower border numbers	upper border numbers
square	1 . . . 10	131 . . . 140
horizontal	11 . . . 18	123 . . . 130
vertical	19 . . . 22	119 . . . 122
total	1 . . . 22	119 . . . 140

Table 30: Numbers for the first border

The next border has size 8 x 12, so both m_2 and n_2 are double-even. Unfortunately, you can't start from the smallest square border of this type as you did with the last border. This border would have size 4, but is not suitable for multi-bordered magic rectangles.

1	13	16	4
12			5
6			11
15	3	2	14

Figure 4.26: Invalid border of size 4 x 4

With such a border, the border numbers must include the ranges $1, 2, \ldots, 6$ and $11, 12, \ldots, 16$ as in figure 4.26, leaving the numbers $7, 8, 9, 10$ for the center. However, not all of the two center rows and columns add up to the same sum. This also means that no bordered magic rectangle $R(4, 4k)$ or $R(4k, 4)$ can exist.

Since such a difference cannot be compensated in the following borders, a square border of size 8 x 8 must be assumed. Suitable borders of this size can easily be determined using the methods for bordered magic squares.[22] In figure 4.36 on page 84 some borders of this size are presented, which of course can still be varied by rotations or reflections.

In this example the square border from figure 4.27 has been chosen, and its larger 14 numbers are incremented by

$$m_2 n_2 - m_1 m_1 = 8 \cdot 12 - 8 \cdot 8 = 32$$

2	64	61	54	12	13	51	3
60							5
6							59
58							7
8							57
9							56
55							10
62	1	4	11	53	52	14	63

2	96	93	86	12	13	83	3
92							5
6							91
90							7
8							89
9							88
87							10
94	1	4	11	85	84	14	95

Figure 4.27: Square border of size 8 x 8 with partially increased numbers

This border is transferred to the target rectangle as usual for double-even sizes, with the right column being moved to the right margin of the target rectangle.

Since the necessary amount of rows has already been reached, only four columns have to be filled with the numbers of a magic rectangle $R(2, 4)$. As always with extensions, the numbers in this rectangle must follow the numbers in the border areas that have been used up to now.

[22] Danielsson [5] see chapter 9.2

2	96	93	86	12	13	83					3
92											5
6											91
90											7
8											89
9											88
87											10
94	1	4	11	85	84	14					95

2	96	93	86	12	13	83	15	81	80	18	3
92											5
6											91
90											7
8											89
9											88
87											10
94	1	4	11	85	84	14	82	16	17	79	95

Figure 4.28: Extension of the square border to the size 8 x 12

The distribution of the border numbers results from table 31.

	lower border numbers	upper border numbers
square	1 . . . 14	83 . . . 96
extension	15 . . . 18	79 . . . 83
total	1 . . . 18	79 . . . 96

Table 31: Numbers for the second border

To be able to insert this border into the target rectangle, all numbers must be increased by the constant c

$$c = m_1 + n_1 - 2 = 10 + 14 - 2 = 22$$

so that they do not overlap with numbers in the first border.

24	118	115	108	34	35	105	37	103	102	40	25
114											27
28											113
112											29
30											111
31											110
109											32
116	23	26	33	107	106	36	104	38	39	101	117

Figure 4.29: Completely filled border of size 8 x 12

In figure 4.30, this border has been inserted into the target rectangle.

1	8	135	136	139	11	12	128	127	126	125	17	18	4
19	24	118	115	108	34	35	105	37	103	102	40	25	122
121	114											27	20
120	28											113	21
22	112											29	119
7	30											111	134
9	31											110	132
131	109											32	10
138	116	23	26	33	107	106	36	104	38	39	101	117	3
137	133	6	5	2	130	129	13	14	15	16	124	123	140

Figure 4.30: Rectangle $R(10, 14)$ with two inserted borders

The third border has size 6 x 10 and can be created as in the first example.

1	8	31	32	35	4
7					30
9					28
27					10
34					3
33	29	6	5	2	36

1	8	55	56	59	4
7					54
9					52
51					10
58					3
57	53	6	5	2	60

1	8	55	56	59					4
7									54
9									52
51									10
58									3
57	53	6	5	2					60

1	8	55	56	59	11	49	48	14	4
7									54
9									52
51									10
58									3
57	53	6	5	2	50	12	13	47	60

Figure 4.31: Border with size 6 x 10

The numbers in the border from figure 4.31 must be increased by the constant c so that numbers in the second border do not appear twice. The constant c is now calculated with

$$c = c + m_2 + n_2 - 2 = 22 + 8 + 12 - 2 = 40$$

1	8	55	56	59	11	49	48	14	4
7									54
9									52
51									10
58									3
57	53	6	5	2	50	12	13	47	60

41	48	95	96	99	51	89	88	54	44
47									94
49									92
91									50
98									43
97	93	46	45	42	90	52	53	87	100

Figure 4.32: Completely filled border of size 6 × 10

If this border is also inserted into the target rectangle, the intermediate result from figure 4.33 is created, which already contains three borders.

1	8	135	136	139	11	12	128	127	126	125	17	18	4
19	24	118	115	108	34	35	105	37	103	102	40	25	122
121	114	41	48	95	96	99	51	89	88	54	44	27	20
120	28	47									94	113	21
22	112	49									92	29	119
7	30	91									50	111	134
9	31	98									43	110	132
131	109	97	93	46	45	42	90	52	53	87	100	32	10
138	116	23	26	33	107	106	36	104	38	39	101	117	3
137	133	6	5	2	130	129	13	14	15	16	124	123	140

Figure 4.33: Rectangle $R(10, 14)$ with three inserted borders

Now only a rectangle $R_4(4, 8)$ in the center is missing, which is created as in the previous example. The increment number here is

$$c = c + m_3 + n_3 - 2 = 40 + 6 + 10 - 2 = 54$$

and the rectangle with the numbers incremented by $c = 54$ is shown in figure 4.34.

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

a) $R_1(2, 8)$

1	31	30	4	5	27	26	8
32	2	3	29	28	6	7	25

b) increased numbers in R_1

1	15	14	4	5	11	10	8
16	2	3	13	12	6	7	9

c) $R_2(2, 8)$

9	23	22	12	13	19	18	16
24	10	11	21	20	14	15	17

d) increased numbers in R_2

1	31	30	4	5	27	26	8
9	23	22	12	13	19	18	16
24	10	11	21	20	14	15	17
32	2	3	29	28	6	7	25

e) R_1 and R_2 inserted into $R_4(4, 8)$

55	85	84	58	59	81	80	62
63	77	76	66	67	73	72	70
78	64	65	75	74	68	69	71
86	56	57	83	82	60	61	79

f) final rectangle $R_4(4, 8)$

Figure 4.34: Rectangle $R_4(4, 8)$ for the center

If you insert the rectangle R_4 into the empty center area of the target rectangle, the multi-bordered magic rectangle from figure 4.35 with row sums 987 and column sums 705 is created.

1	8	135	136	139	11	12	128	127	126	125	17	18	4
19	24	118	115	108	34	35	105	37	103	102	40	25	122
121	114	41	48	95	96	99	51	89	88	54	44	27	20
120	28	47	55	85	84	58	59	81	80	62	94	113	21
22	112	49	63	77	76	66	67	73	72	70	92	29	119
7	30	91	78	64	65	75	74	68	69	71	50	111	134
9	31	98	86	56	57	83	82	60	61	79	43	110	132
131	109	97	93	46	45	42	90	52	53	87	100	32	10
138	116	23	26	33	107	106	36	104	38	39	101	117	3
137	133	6	5	2	130	129	13	14	15	16	124	123	140

Figure 4.35: Multi-bordered magic square of size 10×14

1	7	8	59	60	61	62	2
10							55
11							54
14							51
52							13
53							12
56							9
63	58	57	6	5	4	3	64

1	13	14	53	54	61	62	2
6							59
7							58
10							55
56							9
57							8
60							5
63	52	51	12	11	4	3	64

2	5	8	58	59	61	64	3
9							56
12							53
14							51
52							13
54							11
55							10
62	60	57	7	6	4	1	63

2	6	7	57	60	61	64	3
10							55
11							54
14							51
52							13
53							12
56							9
62	59	58	8	5	4	1	63

3	4	13	54	56	59	63	8
1							64
12							53
14							51
55							10
58							7
60							5
57	61	52	11	9	6	2	62

4	7	10	56	57	59	62	5
2							63
11							54
14							51
52							13
53							12
64							1
60	58	55	9	8	6	3	61

5	2	14	55	56	58	62	8
1							64
11							54
13							52
53							12
59							6
61							4
57	63	51	10	9	7	3	60

5	4	13	54	55	59	62	8
1							64
9							56
14							51
53							12
58							7
63							2
57	61	52	11	10	6	3	60

6	1	4	57	60	62	63	7
10							55
12							53
13							52
51							14
54							11
56							9
58	64	61	8	5	3	2	59

7	1	2	59	60	61	62	8
10							55
11							54
14							51
52							13
53							12
56							9
57	64	63	6	5	4	3	58

8	1	4	55	58	62	63	9
6							59
12							53
13							52
51							14
54							11
60							5
56	64	61	10	7	3	2	57

8	2	3	55	58	61	64	9
6							59
12							53
13							52
51							14
54							11
60							5
56	63	62	10	7	4	1	57

Figure 4.36: Some square border of size 8×8

5 Symmetrical magic rectangles

Similar to the construction method of bordered magic rectangles from chapter 4, I am not aware of any article in the literature that describes the construction of symmetrical (associated) magic rectangles. Therefore, I present my own method here, where the height m and the width n must both be double-even and additionally $m, n \geq 4$ applies.

My construction is based on nesting of arbitrary symmetrical magic 4×4 squares, of which there are $48 \cdot 8 = 384$ in total, including rotations and reflections. Some of the 48 basic squares are shown in figure 5.14 on page 90.

In the first example, a symmetrical rectangle with $m = 4$ rows and $n = 8$ columns will be created, which illustrates the basic idea of this method. Start with any symmetrical 4×4 - magic square, whose upper eight numbers are increased by $m \cdot n - 2 \cdot 8 = 16$, while the lower eight numbers remain unchanged.

3	10	16	5
13	8	2	11
6	15	9	4
12	1	7	14

a) symmetrical

3	26	32	5
29	8	2	27
6	31	25	4
28	1	7	30

b) partially increased numbers

Figure 5.1: First symmetrical initial square with partially increased numbers

Then, the two left columns are moved to the left margin of the target rectangle, as shown in figure 5.2, and the two right columns to the right margin.

3	26					32	5
29	8					2	27
6	31					25	4
28	1					7	30

Figure 5.2: Target rectangle with the first inserted square of size 4×4

A second symmetrical 4×4 - magic square is chosen for the missing center. The numbers of this square are incremented differently. The eight smaller numbers are increased here by 8, while the increment number for the eight larger numbers is reduced by 8 compared to the first step, and thus also amounts to 8.

7	12	2	13
9	6	16	3
14	1	11	8
4	15	5	10

15	20	10	21
17	14	24	11
22	9	19	16
12	23	13	18

Figure 5.3: Second symmetrical initial square with differently increased numbers

If you insert this modified square into the target rectangle, the result is the symmetrical magic rectangle of size 4×8 from figure 5.4 with row sums 132 and column sums 66.

3	26	15	20	10	21	32	5
29	8	17	14	24	11	2	27
6	31	22	9	19	16	25	4
28	1	12	23	13	18	7	30

Figure 5.4: Symmetrical magic rectangle of size 4 x 8

Example 2

In the second example, a rectangle $R(8, 12)$ with $m = 8$ rows and $n = 12$ columns will be created, where the nesting is even more clearly visible. In the first step, as in the previous example, a rectangle $R_1(4, 12)$ is created across the entire width. With $m_1 = 4$ rows and $n_1 = 12$ columns, the increment number for the lower eight numbers for the symmetrical magic square is initially $c_1 = 0$ again, i.e. these numbers remain unchanged. But for the upper eight numbers a different increment number c_2 is valid now.

$$c_2 = m_1 \cdot n_1 - 2 \cdot 8 = 4 \cdot 12 - 16 = 32$$

This results in the square from figure 5.5, whose halves are shifted to the two margins of R_1.

13	3	12	6
2	16	7	9
8	10	1	15
11	5	14	4

45	3	44	6
2	48	7	41
8	42	1	47
43	5	46	4

45	3									44	6
2	48									7	41
8	42									1	47
43	5									46	4

Figure 5.5: First symmetrical initial square with partially increased numbers

For the next square to be inserted, the previous increment number c_1 is increased by 8 and for the larger numbers c_2 is decreased by 8.

$$c_1 = c_1 + 8 = 0 + 8 = 8 \qquad c_2 = c_2 - 8 = 32 - 8 = 24$$

This results in the square of figure 5.6 and both halves are moved to the empty margins of R_1 as usual.

5	4	11	14
16	9	2	7
10	15	8	1
3	6	13	12

13	12	35	38
40	33	10	15
34	39	16	9
11	14	37	36

45	3	13	12					35	38	44	6
2	48	40	33					10	15	7	41
8	42	34	39					16	9	1	47
43	5	11	14					37	36	46	4

Figure 5.6: Second symmetrical initial square with differently increased numbers

Now only a third square of size 4 is missing for the center, where the increment numbers change to

$$c_1 = c_1 + 8 = 8 + 8 = 16 \qquad c_2 = c_2 - 8 = 24 - 8 = 16$$

If you insert this modified square into the rectangle R_1, the result is the symmetrical magic rectangle from figure 5.7 with row sums 294 and column sums 98.

15	4	10	5
6	9	3	16
1	14	8	11
12	7	13	2

31	20	26	21
22	25	19	32
17	30	24	27
28	23	29	18

45	3	13	12	31	20	26	21	35	38	44	6
2	48	40	33	22	25	19	32	10	15	7	41
8	42	34	39	17	30	24	27	16	9	1	47
43	5	11	14	28	23	29	18	37	36	46	4

Figure 5.7: Symmetrical Magic Rectangle $R_1(4, 12)$

The rectangle R_1 is now modified to be copied into the target rectangle. The upper $\frac{m_1 \cdot n_1}{2} = 24$ are increased in this step by

$$d_2 = m \cdot n - 4n = 8 \cdot 12 - 4 \cdot 12 = 48$$

while the lower 24 numbers remain unchanged with $d_1 = 0$.

93	3	13	12	79	20	74	21	83	86	92	6
2	96	88	81	22	73	19	80	10	15	7	89
8	90	82	87	17	78	24	75	16	9	1	95
91	5	11	14	76	23	77	18	85	84	94	4

Figure 5.8: Rectangle $R_1(4, 12)$ with partially increased numbers

In the next step, the rows of this modified rectangle R_1 are transferred to the target rectangle. The upper two rows of R_1 are moved to the top margin and the bottom two rows to the bottom margin.

93	3	13	12	79	20	74	21	83	86	92	6
2	96	88	81	22	73	19	80	10	15	7	89
8	90	82	87	17	78	24	75	16	9	1	95
91	5	11	14	76	23	77	18	85	84	94	4

Figure 5.9: Target rectangle with the inserted rectangle $R_1(4, 12)$

Now only another rectangle $R_2(4, 12)$ is missing, which is created like R_1. For the increment numbers d_1 and d_2 applies that they are increased or decreased with each further rectangle $R_i(4, n)$ by $2n$.

$$d_1 = d_1 + 2n = 0 + 2 \cdot 12 = 24 \qquad d_2 = d_2 - 2n = 48 - 24 = 24$$

5	4	10	37	23	30	18	27	40	11	48	41
43	46	35	16	28	17	29	24	13	34	2	7
42	47	15	36	25	20	32	21	33	14	3	6
8	1	38	9	22	31	19	26	12	39	45	44

29	28	34	61	47	54	42	51	64	35	72	65
67	70	59	40	52	41	53	48	37	58	26	31
66	71	39	60	49	44	56	45	57	38	27	30
32	25	62	33	46	55	43	50	36	63	69	68

Figure 5.10: Symmetrical magic rectangle $R_2(4, 12)$ with differently increased numbers

Finally, the rectangle R_2 is inserted into the empty area of the target rectangle and the symmetrical magic rectangle from figure 5.11 with row sums 582 and column sums 388 is created.

93	3	13	12	79	20	74	21	83	86	92	6
2	96	88	81	22	73	19	80	10	15	7	89
29	28	34	61	47	54	42	51	64	35	72	65
67	70	59	40	52	41	53	48	37	58	26	31
66	71	39	60	49	44	56	45	57	38	27	30
32	25	62	33	46	55	43	50	36	63	69	68
8	90	82	87	17	78	24	75	16	9	1	95
91	5	11	14	76	23	77	18	85	84	94	4

Figure 5.11: Symmetrical magic rectangle of size 8 x 12

This method can be extended to rectangles of all sizes, whose width and the height are double-even. A symmetrical magic rectangle of size 12 x 20 is shown in figure 5.12. The row sums are 2410 and the column sums are 1446.

236	6	230	227	224	217	213	212	208	35	34	205	27	30	21	20	15	10	1	239
7	233	9	16	18	23	26	31	201	38	39	204	216	209	219	222	228	229	238	4
197	43	52	53	179	64	173	172	79	161	166	76	66	71	182	57	185	192	196	46
48	194	191	186	58	181	67	70	74	168	163	77	176	169	63	180	54	51	41	199
159	86	90	152	101	139	107	130	114	119	125	124	110	135	100	142	149	91	81	156
84	153	147	93	138	104	136	109	128	121	115	118	129	108	143	97	96	146	158	87
154	83	95	145	144	98	133	112	123	126	120	113	132	105	137	103	148	94	88	157
85	160	150	92	99	141	106	131	117	116	122	127	111	134	102	140	89	151	155	82
42	200	190	187	61	178	72	65	164	78	73	167	171	174	60	183	55	50	47	193
195	45	49	56	184	59	170	175	165	75	80	162	69	68	177	62	188	189	198	44
237	3	12	13	19	22	32	25	37	202	203	40	210	215	218	223	225	232	8	234
2	240	231	226	221	220	211	214	36	207	206	33	29	28	24	17	14	11	235	5

Figure 5.12: Symmetrical magic rectangle of size 12 x 20

However, the basic idea of this construction cannot be transferred to rectangles, whose width or height is single-even. Although such rectangles exist, as the symmetrical magic rectangle in figure 5.13, but they cannot be nested. The required different increments of the lower and upper numbers generates different row or column sums.

9	14	20	18	10	4	→ 75
12	19	1	3	23	17	→ 75
8	2	22	24	6	13	→ 75
21	15	7	5	11	16	→ 75
↓ 50	↓ 50	↓ 50	↓ 50	↓ 50	↓ 50	

a) symmetrical magic rectangle

21	38	44	42	22	16	→ 183
24	43	13	15	47	41	→ 183
20	14	46	48	18	37	→ 183
45	39	19	17	23	40	→ 183
↓ 110	↓ 134	↓ 122	↓ 122	↓ 110	↓ 134	

b) differently increased numbers

Figure 5.13: Increasing numbers in a symmetrical rectangle of size 4 x 6

On the other hand, symmetrical magic rectangles, where both, the width and the height, are single-even, do not exist at all.

1	12	14	7
8	13	11	2
15	6	4	9
10	3	5	16

1	12	15	6
14	7	4	9
8	13	10	3
11	2	5	16

1	12	15	6
8	13	10	3
14	7	4	9
11	2	5	16

1	14	12	7
15	4	6	9
8	11	13	2
10	5	3	16

1	14	15	4
12	7	6	9
8	11	10	5
13	2	3	16

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

1	15	12	6
14	4	7	9
8	10	13	3
11	5	2	16

1	15	12	6
8	10	13	3
14	4	7	9
11	5	2	16

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

1	15	14	4
8	10	11	5
12	6	7	9
13	3	2	16

2	11	16	5
13	8	3	10
7	14	9	4
12	1	6	15

2	11	16	5
7	14	9	4
13	8	3	10
12	1	6	15

2	13	16	3
11	8	5	10
7	12	9	6
14	1	4	15

2	13	16	3
7	12	9	6
11	8	5	10
14	1	4	15

2	16	11	5
13	3	8	10
7	9	14	4
12	6	1	15

2	16	11	5
7	9	14	4
13	3	8	10
12	6	1	15

2	16	13	3
11	5	8	10
7	9	12	6
14	4	1	15

2	16	13	3
7	9	12	6
11	5	8	10
14	4	1	15

3	10	13	8
16	5	2	11
6	15	12	1
9	4	7	14

3	10	13	8
6	15	12	1
16	5	2	11
9	4	7	14

3	13	10	8
16	2	5	11
6	12	15	1
9	7	4	14

3	13	10	8
6	12	15	1
16	2	5	11
9	7	4	14

4	14	9	7
15	1	6	12
5	11	16	2
10	8	3	13

4	14	9	7
5	11	16	2
15	1	6	12
10	8	3	13

4	15	9	6
14	1	7	12
5	10	16	3
11	8	2	13

4	9	14	7
15	6	1	12
5	16	11	2
10	3	8	13

4	9	14	7
5	16	11	2
15	6	1	12
10	3	8	13

5	10	11	8
16	3	2	13
4	15	14	1
9	6	7	12

5	10	11	8
4	15	14	1
16	3	2	13
9	6	7	12

5	11	10	8
16	2	3	13
4	14	15	1
9	7	6	12

5	11	10	8
4	14	15	1
16	2	3	13
9	7	6	12

6	12	9	7
15	1	4	14
3	13	16	2
10	8	5	11

6	12	9	7
3	13	16	2
15	1	4	14
10	8	5	11

6	9	12	7
15	4	1	14
3	16	13	2
10	5	8	11

6	9	12	7
3	16	13	2
15	4	1	14
10	5	8	11

Figure 5.14: Some symmetrical 4 x 4 - squares

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Note

Questions, comments, additions, and corrections to this document are welcome, as are bug reports. Simply send an email to 'dani [[at]] magic-squares dot info'.